Karlsruhe Institute of Technology

Institute for Theoretical Condensed Matter physics

Condensed Matter Theory I WS 2022/2023

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Sheet 13

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Category A

1. Phonon-mediated interaction of electrons, Schrieffer-Wolf-transformation $(5+5+5+15=30$ points)

In this task, we propose to show that the electron-phonon interaction induces an effective electron-electron interaction using the canonical transformation (Schrieffer-Wolf transformation)

(a) A canonical transformation of an operator H is defined by

$$\tilde{H} = e^{-S} H e^{S}$$

Show that, up to a second order in S, the above transformation is equivalent to

$$\tilde{H} = H + [H, S] + \frac{1}{2}\left[[H, S], S\right] + \mathcal{O}(S^3)$$

(b) Now, we consider a Hamiltonian H_0 with a small perturbation V

$$H = H_0 + V$$

The idea of the canonical transformation is to choose the operator S such that the modified Hamiltonian $\tilde{H}$ does not contain the terms linear with respect to V. For $S \ll H_0$, show that this condition is equivalent to

$$V + [H_0, S] = 0$$

(c) Using the eigenstates $|n\rangle$ of the unperturbed Hamiltonian H_0, show that S can be written as

$$\langle n|S|m\rangle = \frac{\langle n|V|m\rangle}{E_m - E_n}$$

where $E_n = \langle n| H_0 |n\rangle$ is the energy of the state $|n\rangle$ corresponding to the unperturbed Hamiltonian. Show that for the modified Hamiltonian, now, holds the equality:

$$\tilde{H} = H_0 + \frac{1}{2}[V, S] + O(V^3)$$

(d) Let's now consider the Frölich Hamiltonian

$$H_{e-ph} = \sum_{p,q,\sigma} V(q) c^{\dagger}_{p+q,\sigma} c_{p,\sigma} \left(a_q + a^{\dagger}_{-q} \right),$$

which describes the electron-phonon interaction, as a perturbation for the Hamiltonian

$$H_0 = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} a^{\dagger}_{\mathbf{q}} a_{\mathbf{q}},$$

with $c(c^{\dagger})$ are the annihilation (creation) operators for electrons, $a(a^{\dagger})$ are phonons operators, $(a_q + a^{\dagger}_{-q}) \propto u(q)$ where u is the displacement of the ions, and $\omega_{-q} = \omega_q$. Using a canonical transformation, derive the effective electron-electron Hamiltonian at zero T. To do this, evaluate the matrix elements of S. Since we are only interested in the low-temperature behavior, the relevant matrix elements in S are determined by transitions between states $|0\rangle_{ph}|1_{\mathbf{p}}\rangle_e$ (zero phonons and an electron with the momentum $\mathbf{p}$) and $|1_{\mathbf{q}}\rangle_{ph}|1_{\mathbf{p-q}}\rangle_e$ (extra phonon with a momentum $\mathbf{q}$, and a fermion with the momentum $\mathbf{p} - \mathbf{q}$). Rewrite S in the form $S = \sum_{p,q,\sigma} V(q) c^{\dagger}_{p+q,\sigma} c_{p,\sigma} \left(\alpha_{p,q} a_q + \beta_{p,q} a^{\dagger}_{-q} \right)$ which is true for low-excitation limit of low T. Then, the matrix elements with respect to phonons, $\langle 1_q | [H_{e-ph}, S] | 0 \rangle_{ph}$, for the correction to the unperturbed Hamiltonian ($[H_{e-ph}, S]$) can be considered as an effective electron-electron interaction.

Category B

2. Cooper problem (20 Points)

It was shown in the lecture that electrons above the Fermi sea can form bounded pairs even for vanishingly small attractive interactions. We extend this example taking into account holes excitations below the Fermi level.

The electron-electron interaction is reduced to

$$g_{k,q} = \begin{cases} -g & |\epsilon_k - \epsilon_{k-q}| \leqslant \omega_D \\ 0 & |\epsilon_k - \epsilon_{k-q}| > \omega_D \end{cases},$$

i.e., in the interval of the width $2\omega_D$ the interaction constant and attractive, and vanishes beyond. Dispersion of the quasiparticles in the vicinity of the Fermi level is linearly approximated.

Analogous to the lecture, calculate the energy of the state with two quasiparticles (electrons or holes) and find the binding energy Δ per quasiparticle.