Karlsruhe Institute of Technology
Institute for Theoretical Condensed Matter physics

**Condensed Matter Theory I WS 2022/2023**

**Prof. Dr. A. Shnirman**
**Dr. D. Shapiro, Dr. H. Perrin**

**Sheet 14**
**Tutorial: 16.02.2023**

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## Category A

**1. Properties of the BCS ground state** (5 + 15 + 10 + 5 = 35 points)

The BCS ground state $|\Phi_{\mathrm{BCS}}\rangle$ was derived in the lecture.

(a) Show that the ground state is properly normalized, $\langle \Phi_{\mathrm{BCS}} | \Phi_{\mathrm{BCS}} \rangle = 1$.

(b) Calculate the expectation value of the electron number operator $N = \sum_{\boldsymbol{k}\sigma} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}$ and its standard deviation in the ground state.

(c) Let us define the operator of Cooper-pair creation, $B^\dagger_{\boldsymbol{k}} = c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}$ (Not to be confused with the Bogoliubov operator $b_{\boldsymbol{k}}$!). Calculate the expectation value $< B^\dagger_{\boldsymbol{k}} >$ in the BCS ground state. Show that $< B^\dagger_{\boldsymbol{k}} >$ as a function of $k$ has a maximum at the Fermi momentum ($\Delta(\boldsymbol{k}) \equiv \Delta \in \mathbb{R}$ for simplicity).

(d) Calculate the commutators $[B_{\boldsymbol{k}}, B^\dagger_{\boldsymbol{k}'}]$, $[B_{\boldsymbol{k}}, B_{\boldsymbol{k}'}]$, and $[B^\dagger_{\boldsymbol{k}}, B^\dagger_{\boldsymbol{k}'}]$ and their ground-state expectation values. Decide whether Cooper pairs are bosons.

## Category B

**2. 4-component Nambu spinor with spin-orbit coupling** (15 points)

Let us generalize the $2 \times 2$ Nambu matrix formalism (see lecture notes) to explicitly spin-dependent cases, where $\xi_{\boldsymbol{k}}$ in the mean-field Hamiltonian is replaced by $h_0(\boldsymbol{k}) = \xi_{\boldsymbol{k}} + \mathbf{b}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}$ with a vector $\mathbf{b}_{\boldsymbol{k}}$. Thus, $h_0$ is a $2\times2$ matrix in spin space. Let us define the 4-Nambu spinor $\left(c^\dagger_{k,\uparrow}, c^\dagger_{k,\downarrow}, c_{-k,\downarrow}, -c_{-k,\uparrow}\right)$. Find the corresponding $4\times4$ Hamiltonian.
*Remark: Note that this Hamiltonian is redundant. If terms can be expressed through more than one matrix element, distribute them evenly between these elements.*