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Condensed Matter Theory 1 — Exercise 9

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To be discussed on: *Thursday 2024/01/11*

1. Conductivity and Hall effect

In the relaxation time approximation, the linearized Boltzmann equation in the presence of both an electric and magnetic field, $\boldsymbol{E}$ and $\boldsymbol{B}$, reads

$$\frac{\partial}{\partial t}\delta f_{\boldsymbol{k}}(t) - e\frac{\partial f^{(0)}_{\boldsymbol{k}}}{\partial \epsilon_{\boldsymbol{k}}}\boldsymbol{v}_{\boldsymbol{k}} \cdot \boldsymbol{E} - e\left(\frac{1}{c}\boldsymbol{v}_{\boldsymbol{k}} \times \boldsymbol{B}\right)\nabla_{\boldsymbol{k}}\delta f_{\boldsymbol{k}}(t) = -\frac{\delta f_{\boldsymbol{k}}(t)}{\tau} \ . \tag{1}$$

Here, $-e < 0$ is the electron charge. The kinetic energy of electrons is given as $\epsilon_{\boldsymbol{k}} = k^2/(2m)$ and $\boldsymbol{v}_{\boldsymbol{k}} = \boldsymbol{k}/m$. We set $\hbar = 1$. Moreover, assume that the relaxation time τ is independent of $\boldsymbol{k}$. In equilibrium, the distribution is given by the Fermi-Dirac distribution $f^{(0)}_{\boldsymbol{k}} = 1/(1+\exp((\epsilon_{\boldsymbol{k}} - \mu)/T))$ and $\delta f_{\boldsymbol{k}}(t) = f_{\boldsymbol{k}}(t) - f^{(0)}_{\boldsymbol{k}}$.

a) Determine the static solution with $\delta f_{\boldsymbol{k}}(t) \equiv \delta f_{\boldsymbol{k}}$ such that $\frac{\partial}{\partial t}\delta f_{\boldsymbol{k}}(t) = 0$. To find the static solution, consider the ansatz

$$\delta f_{\boldsymbol{k}} = -\frac{\partial f^{(0)}_{\boldsymbol{k}}}{\partial \epsilon_{\boldsymbol{k}}}\boldsymbol{k} \cdot \boldsymbol{a} \ . \tag{2}$$

Show that using this ansatz in the Boltzmann equation yields the following condition for the vector $\boldsymbol{a}$:

$$\boldsymbol{a} - \frac{e\tau}{mc}\boldsymbol{B} \times \boldsymbol{a} = -\frac{e\tau}{m}\boldsymbol{E} \ . \tag{3}$$

This linear system determines the coefficients of the vector $\boldsymbol{a}$. Show that its solution is given by

$$\boldsymbol{a} = -\frac{e\tau}{m}\frac{1}{1+\omega_c^2\tau^2}\left(\boldsymbol{E} + \omega_c\tau\hat{\boldsymbol{B}} \times \boldsymbol{E} + \omega_c^2\tau^2\left(\hat{\boldsymbol{B}} \cdot \boldsymbol{E}\right)\hat{\boldsymbol{B}}\right) \ , \tag{4}$$

with the cyclotron frequency $\omega_c = Be/(mc)$ and $\hat{\boldsymbol{B}} = \boldsymbol{B}/B$.

b) The electrical current is defined as

$$\boldsymbol{j} = 2(-e)\int \frac{d^3\boldsymbol{k}}{(2\pi\hbar)^3}\boldsymbol{v}_{\boldsymbol{k}} f_{\boldsymbol{k}} = -2e\int \frac{d^3\boldsymbol{k}}{(2\pi\hbar)^3}\boldsymbol{v}_{\boldsymbol{k}}\,\delta f_{\boldsymbol{k}}. \tag{5}$$

Calculate $\boldsymbol{j}$ with the help of Eq. (4).
Hint:

$$\int d^3\boldsymbol{k}\left(-\frac{\partial f^{(0)}_{\boldsymbol{k}}}{\partial \epsilon_{\boldsymbol{k}}}\right)\boldsymbol{k}\,(\boldsymbol{k} \cdot \boldsymbol{A}) = 4\pi^3\hbar^3 mn\,\boldsymbol{A} \quad \text{for all vectors } \boldsymbol{A} \tag{6}$$

with the electron density n.

c) For a magnetic field in the z-direction, determine the conductivity tensor $\boldsymbol{\sigma}_H(\boldsymbol{B})$ from the relation $\boldsymbol{j} = \boldsymbol{\sigma}_H(\boldsymbol{B})\,\boldsymbol{E}$. Compare with the known result of classical Drude theory.

d) Sketch the experimental setup which leads to the definition of the Hall coefficient $R_H = \left.\frac{E_y}{j_x B}\right|_{j_y=0}$. Calculate the Hall coefficient R_H.

2. Conservation laws

In the absence of external fields, the Boltzmann equation takes the form

$$\frac{\partial}{\partial t} f_{\boldsymbol{k}}(t) = \mathcal{I}_{\boldsymbol{k}}\left(\{f\}\right), \tag{7}$$

with the collision integral $\mathcal{I}_{\boldsymbol{k}}$. In the following exercises, consider the particle number N, total kinetic energy E, and total momentum $\boldsymbol{P}$, given by

$$\begin{aligned} N &= V \int \frac{d^3\boldsymbol{k}}{(2\pi\hbar)^3} f_{\boldsymbol{k}}(t) \\ E &= V \int \frac{d^3\boldsymbol{k}}{(2\pi\hbar)^3} \epsilon_{\boldsymbol{k}} f_{\boldsymbol{k}}(t) \;, \\ \boldsymbol{P} &= V \int \frac{d^3\boldsymbol{k}}{(2\pi\hbar)^3} \boldsymbol{k} f_{\boldsymbol{k}}(t) \end{aligned} \tag{8}$$

and check whether they are conserved quantities for specific examples of the collision integral.

a) *Electron-electron interaction* — The collision integral for fermions with the Coulomb interaction $U(q) = e^2/q^2$ reads

$$\begin{aligned} \mathcal{I}^{ee}_{\boldsymbol{k}}\left(\{f\}\right) =& \frac{\pi}{\hbar} V^2 \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} \frac{d^3\boldsymbol{q}}{(2\pi\hbar)^3} |U(q) - U(|\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{k}|)|^2 \delta\left(\epsilon_{\boldsymbol{k}} + \epsilon_{\boldsymbol{p}} - \epsilon_{\boldsymbol{k}+\boldsymbol{q}} - \epsilon_{\boldsymbol{p}-\boldsymbol{q}}\right) \cdot \\ & \cdot \Big[(1-f_{\boldsymbol{k}})(1-f_{\boldsymbol{p}}) f_{\boldsymbol{k}+\boldsymbol{q}} f_{\boldsymbol{p}-\boldsymbol{q}} - f_{\boldsymbol{k}} f_{\boldsymbol{p}} (1-f_{\boldsymbol{k}+\boldsymbol{q}})(1-f_{\boldsymbol{p}-\boldsymbol{q}})\Big] \end{aligned} \tag{9}$$

with the volume V. Show that N, E, and $\boldsymbol{P}$ are conserved, i.e., show that $\frac{dN}{dt} = 0$, $\frac{dE}{dt} = 0$, $\frac{d\boldsymbol{P}}{dt} = 0$.

b) *Relaxation time approximation* — In this approximation, the collision integral is approximated as

$$\mathcal{I}^{\tau}_{\boldsymbol{k}} = -\frac{f_{\boldsymbol{k}}(t) - f^{(0)}_{\boldsymbol{k}}}{\tau_{\boldsymbol{k}}}, \tag{10}$$

with the stationary equilibrium distribution $f^{(0)}_{\boldsymbol{k}}$. Solve the Boltzmann equation in this approximation. Argue that, in general, the solution violates the conservation of N, E and $\boldsymbol{P}$.

c) *Scattering from impurities* — Consider fermions in a disordered medium where they scatter from impurities with the matrix elements $U_{imp}(\boldsymbol{k},\boldsymbol{p}) = \langle\boldsymbol{k}|\,\hat{H}_{imp}\,|\boldsymbol{p}\rangle = \langle\boldsymbol{p}|\,\hat{H}_{imp}\,|\boldsymbol{k}\rangle^* = U^*_{imp}(\boldsymbol{p},\boldsymbol{k})$. The corresponding collision integral reads

$$\mathcal{I}^{imp}_{\boldsymbol{k}}\left(\{f\}\right) = \frac{2\pi}{\hbar} V \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} |U_{imp}(\boldsymbol{k},\boldsymbol{p})|^2 \delta(\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{p}})(f_{\boldsymbol{p}} - f_{\boldsymbol{k}}) \;. \tag{11}$$

Show that N and E are conserved but $\boldsymbol{P}$ is not.