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Condensed Matter Theory 1 — Exercise 10

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To be discussed on: *Thursday 2024/01/18*

1. The anomalous skin effect

The *skin effect* describes the limited penetration depth δ of an electromagnetic wave into a metal. Under the assumption that the electric field $\boldsymbol{E}$ is approximately uniform on the scale of the electron's mean free path ℓ, i.e., assuming $\delta \gg \ell$, one can derive that the penetration depth is approximately given as

$$\delta \approx \frac{c}{\sqrt{2\pi\sigma_0\omega}}, \tag{1}$$

where $\sigma_0 = ne^2\tau/m$ is the Drude conductivity, c is the speed of light, and ω is the frequency of the wave. The dependence $\delta \propto 1/\sqrt{\omega}$ is characteristic of the *normal* skin effect.
However, for clean metals at low temperature, both σ_0 and ℓ are large. If the frequency ω is also large enough then from Eq. (1) we note that δ becomes very small and, in particular, smaller than ℓ. Hence, in this limit, Eq. (1) does not hold anymore. Instead, for $\delta \ll \ell$ the so-called *anomalous* skin effect determines the penetration depth δ. The goal of this exercise is to derive an expression for the anomalous skin effect using the linearized Boltzmann equation.

As a first qualitative estimate, consider that only electrons which propagate their whole mean free path within the skin layer are significantly influenced by the electric field $\boldsymbol{E}$. For these electrons $\mathrm{d}\theta \sim \delta/\ell$, see Fig. 1. For the number of electrons with momenta directed in a certain solid angle element $\mathrm{d}\Omega$ one obtains an effective electron density

$$n_{\text{eff}} \sim n\frac{\mathrm{d}\Omega}{4\pi} \sim n\frac{\delta}{\ell}\ . \tag{2}$$

The effective conductivity follows as $\sigma_{\text{eff}} = n_{\text{eff}}e^2\tau/m = \alpha\sigma_0\delta/\ell$, where α is an unknown factor.

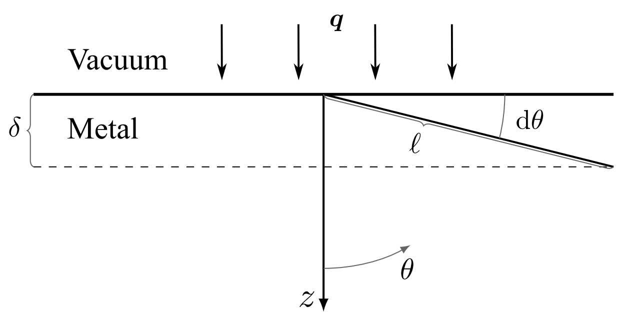

Figure 1: Cross-section of the metal surface. Vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to z.

a) Derive the penetration depth δ by replacing σ_0 with σ_{eff} in Eq. (1). How is δ related to ω in this case?

A more elaborate calculation can be done by using the *linearized Boltzmann equation* with an electric field $\boldsymbol{E}$

$$\partial_t\delta f - e\frac{\partial f_0}{\partial \epsilon_{\boldsymbol{k}}}\boldsymbol{v}_{\boldsymbol{k}}\cdot\boldsymbol{E} + \boldsymbol{v}_{\boldsymbol{k}}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\delta f = -\frac{\delta f}{\tau}\ , \tag{3}$$

where $\delta f \equiv \delta f(\boldsymbol{r}, \boldsymbol{k}, t)$ is the deviation from the equilibrium distribution $f_0(\boldsymbol{k})$ and $\boldsymbol{v_k} = \frac{\hbar}{m}\boldsymbol{k}$.

b) Solve Eq. (3) for the Fourier transform $\delta f(\boldsymbol{q}, \boldsymbol{k}, \omega)$ with $\delta f(\boldsymbol{q}, \boldsymbol{k}, \omega) = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{r}\, \delta f(\boldsymbol{r}, \boldsymbol{k}, t) e^{-i(\boldsymbol{qr}-\omega t)}$. Use the relation for the electrical current density $\boldsymbol{j} = 2(-e)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \boldsymbol{v_k} \delta f$ to obtain the components of the conductivity tensor σ defined via $j_\alpha = \sigma_{\alpha\beta} E_\beta$. Show that

$$\sigma_{\alpha\beta}(\boldsymbol{q}, \omega) = 2e^2 \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \left(-\frac{\partial f_0}{\partial \epsilon_{\boldsymbol{k}}}\right) \frac{v_{\boldsymbol{k}}^\alpha v_{\boldsymbol{k}}^\beta}{\frac{1}{\tau} + i(\boldsymbol{q v_k} - \omega)} \ . \tag{4}$$

Here $\alpha, \beta \in \{x, y, z\}$.

In the following, consider the case $\boldsymbol{q} || \hat{\boldsymbol{z}}$ shown in Fig. 1.

c) Show that all non-diagonal components of $\sigma_{\alpha\beta}$ are zero and that $\sigma_{xx} = \sigma_{yy}$.

d) Consider the free-electron approximation and the zero temperature limit $T \to 0$:

$$\boldsymbol{v_k} = \frac{\hbar}{m}\boldsymbol{k}, \quad \epsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m}, \quad \left(-\frac{\partial f_0}{\partial \epsilon_{\boldsymbol{k}}}\right) = \delta(\epsilon - \epsilon_F), \quad \epsilon_F = \frac{\hbar^2 k_F^2}{2m}, \quad k_F = (3\pi^2 n)^{1/3}.$$

Show that in this case

$$\sigma_{xx}(\boldsymbol{q}, \omega) = \frac{e^2}{4\pi^2 m} k_F^3 \int_{-1}^{1} \mathrm{d}(\cos\theta)\, \frac{\sin^2\theta}{\tau^{-1} - i(\omega - q v_F \cos\theta)} \ . \tag{5}$$

Convince yourself that in the limit $q = 0$ Eq. (5) reduces to the optical Drude conductivity $\sigma_{xx} = \sigma_0/(1 - i\tau\omega)$.

e) In the limit $\tau \to \infty$ and $q v_F \gg \omega$, which is relevant for the anomalous skin effect, show that

$$\sigma(q)_{xx} \approx \frac{3\pi}{4} \frac{\sigma_0}{\tau q v_F} \ . \tag{6}$$

Hint: Use the Sokhotski-Plemelj theorem $\lim_{\eta \to 0^+} \frac{1}{x \pm i\eta} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x)$, *where* $\mathcal{P}$ *denotes the Cauchy principal value.*
Note for the German students: You may know this as the Dirac identity.

The dynamics of the electric field are described by the Maxwell equations. For an incoming electromagnetic wave directed in z-direction with an oscillating electric field in x-direction, we obtain the differential equation

$$\partial_z^2 E_x - \frac{1}{c^2}\partial_t^2 E_x = \frac{4\pi}{c^2}\partial_t j_x \ . \tag{8}$$

Note that the electric field profile depends on the current density j_x which itself depends on the electric field profile and the conductivity tensor, as derived in the previous tasks in Fourier space. However, we cannot simply Fourier transform Eq. (8) due to the material parameter jump at $z = 0$, see Fig. 2.
To solve this problem, we use the following trick: We can replace the vacuum half-space by the mirror image of the metallic half-space. Thereby, we effectively obtain homogeneously metallic sample with the only exception being at $z = 0$ where an additional delta-function encodes the boundary conditions of the original setup, see Fig. 2. However, since we are not interested in the precise value of the field but in its decay length, also the details of the boundary condition are not important and it is sufficient to introduce them with a generic factor. Thus, in applying this mirror trick we have to add a term $+A\delta(z)$ to the right hand side of Eq. (8).

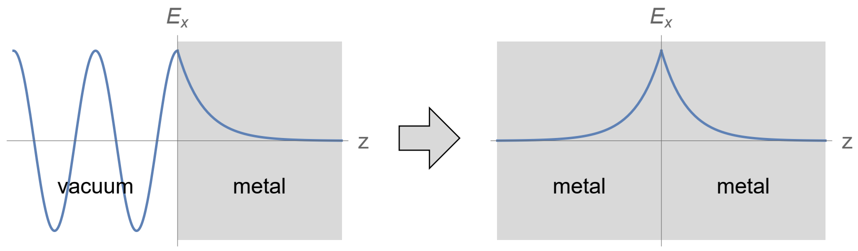

Figure 2: Left: Sketch of the electric field profile in vaccum and in the metal. Right: Field profile in ficticious metallic sample, obtained by extending the original field profile in the metallic half-space via a mirror operation into the vacuum half-space.

f) Show that the electric field profile within the metal can be brought to the following form

$$E_x^0(z) = -A \int_{-\infty}^{\infty} \mathrm{d}q \frac{e^{iqz}}{q^2 - \frac{\omega^2}{c^2} - i\frac{4\pi}{c^2}\sigma(q)\omega}. \tag{9}$$

g) Substitute (6) into (9) and neglecting the term ω^2/c^2 in denominator of the integrand in (9) show that the penetration depth is

$$\delta \approx \left(\frac{\tau c^2 v_F}{3\pi^2 \sigma_0 \omega}\right)^{1/3}. \tag{10}$$

Hint: Use the residue theorem.