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Condensed Matter Theory 1 — Exercise 13

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https://ilias.studium.kit.edu/goto.php?target=crs_2219528
To be discussed on: *Thursday 2024/02/08*

1. Fermionic Bogoliubov transformation

In the context of the BCS theory, we encountered a Hamiltonian with *anomalous* terms consisting of two fermionic creation and annihilation operators. Such terms violate the $U(1)$ gauge symmetry of the Hamiltonian. Such Hamiltonians can be diagonalized with the help of a Bogoliubov transformation, which will be considered in some detail in this exercise.

a) Consider fermionic creation and annihilation operators $c_\sigma^\dagger$ and c_σ, where spin up ($\uparrow$) or down($\downarrow$) is denoted as $\sigma = +1$ or $\sigma = -1$, respectively. New fermionic operators a_σ and $a_\sigma^\dagger$ can be introduced via the Bogoliubov transformation

$$c_\sigma = \alpha_\sigma a_\sigma + \beta_\sigma a_{-\sigma}^\dagger \quad \text{and} \quad c_\sigma^\dagger = \alpha_\sigma^* a_\sigma^\dagger + \beta_\sigma^* a_{-\sigma} \ , \tag{1}$$

or, equivalently, for the Nambu spinor

$$\vec{c} \equiv \begin{pmatrix} c_\uparrow \\ c_\downarrow^\dagger \end{pmatrix} = U \begin{pmatrix} a_\uparrow \\ a_\downarrow^\dagger \end{pmatrix} \equiv U \vec{a} \ , \tag{2}$$

where the matrix U is simply related to the coefficients α_σ and β_σ. By using the fermionic operator algebra, show that this matrix U is a unitary matrix, i.e., $U^\dagger U = \mathbb{1}$.

b) Consider a system of fermions described by the Hamiltonian

$$\mathcal{H} = \varepsilon \left(c_\uparrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\downarrow \right) - \Delta c_\uparrow^\dagger c_\downarrow^\dagger - \Delta^* c_\downarrow c_\uparrow \ , \quad \varepsilon \in \mathbb{R}, \Delta \in \mathbb{C} \ . \tag{3}$$

Using Nambu spinors rewrite the Hamiltonian in matrix form, $\mathcal{H} = \frac{1}{2}\vec{c}^\dagger H \vec{c} + h$ and determine the matrix H and the constant h. Diagonalize the matrix H with the help of the Bogoliubov transformation and compute the eigenenergies.
Bonus: The eigenenergies come in pairs $\pm E$. This is related to the particle-hole symmetry $\tau^y H^* \tau^y = -H$, where τ^y is a Pauli matrix. Why?

2. Bosonic Bogoliubov transformation

In this exercise, we consider bosonic operators and examine the bosonic version of a Bogoliubov transformation.

a) Consider bosonic creation and annihilation operators $d^\dagger$ and d. New bosonic operators are introduced via the bosonic Bogoliubov transformation

$$d = \alpha b + \beta b^\dagger \quad \text{and} \quad d^\dagger = \alpha^* b^\dagger + \beta^* b \ , \tag{4}$$

or, equivalently,

$$\vec{d} \equiv \begin{pmatrix} d \\ d^\dagger \end{pmatrix} = B \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \equiv B\vec{a} \ , \tag{5}$$

where the matrix B is related to the coefficients α and β. Using the bosonic operator algebra, show that the matrix B fulfils the relation $B^\dagger \tau^z B = \tau^z$, where τ^z is the Pauli matrix, i.e., B is not unitary.

b) Consider the Hamiltonian

$$\mathcal{H} = \varepsilon d^\dagger d - \frac{\Delta}{2} d^\dagger d^\dagger - \frac{\Delta^*}{2} dd \ , \quad \varepsilon \in \mathbb{R}, \Delta \in \mathbb{C} \ . \tag{6}$$

Rewrite the Hamiltonian in matrix form, $\mathcal{H} = \frac{1}{2}\vec{d}^\dagger H \vec{d} + h$, and determine the matrix H and the constant h. Diagonalize the matrix H with the help of the bosonic Bogoliubov transformation and determine the eigenenergies.
Hint: The diagonalization requires to choose the coefficients of the matrix B such that $B^\dagger H B = D$ becomes a diagonal matrix D. Alternatively, you might want to use that $\tilde{B}^{-1}\tau^z H \tilde{B} = D\tau^z$ where $\tilde{B} = B\tau^z$ and $\tilde{B}^{-1} = B^\dagger \tau^z$.
Bonus: The diagonal matrix D is proportional to the unit matrix $D = E\mathbb{1}$. Why?

3. Superconductors and Coulomb interaction

In this exercise we investigate a minimal extension to the BCS theory of superconductivity that was discussed in the lecture. As a generalization of the BCS model we introduce a momentum dependent coupling V_{kp} and consider the Hamiltonian

$$H = \sum_{k,\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k,k'} V_{k'k} c^\dagger_{-k'\downarrow} c^\dagger_{k'\uparrow} c_{k\uparrow} c_{-k\downarrow}.$$

a) Apply a mean-field approximation to the above Hamiltonian and write it in terms of the now momentum dependent order parameter

$$\Delta_k = \sum_p V_{kp} \langle c_{p\uparrow} c_{-p\downarrow} \rangle.$$

and the Nambu spinors $\vec{\psi}_k = (c_{k\uparrow}, c^\dagger_{-k\downarrow})^T$. Calculate its eigenenergies. Before you start, convince yourself that for H to be hermitian it must hold $V^*_{kp} = V_{pk}$.

b) In the lecture, the gap equation was derived self-consistently for $T \neq 0$. Repeat this derivation for the new Hamiltonian and show that the gap equation has the form

$$\Delta_k = -\int \frac{d^3p}{(2\pi)^3} V_{kp} \frac{\tanh\left(\frac{E_p}{2k_BT}\right)}{2E_p} \Delta_p,$$

where E_p is the eigenenergy of the mean-field Hamiltonian.

c) In the BCS model the coupling V_{kp} was approximated by a constant attractive potential below the Debye frequency ($|\xi_k|, |\xi_p| < \omega_D$). In the lecture, this was motivated by overscreening from electron-phonon interaction. However, revisiting the results from the lecture you may notice that above ω_D the effective potential is still repulsive. Therefore, we introduce a repulsive interaction V, representing Coulomb repulsion, below some large cut-off frequency $\omega_C > \omega_D$, such that

$$V_{kp} = -g\Theta(\omega_D - |\xi_k|)\Theta(\omega_D - |\xi_p|) + V\Theta(\omega_C - |\xi_k|)\Theta(\omega_C - |\xi_p|)$$

where Θ is the Heavyside-function. It is then convenient to write the gap, which is of course still constant on the respective intervals, as

$$\Delta_k = \begin{cases} \Delta_< & \text{if } |\xi_k| < \omega_D, \\ \Delta_> & \text{if } \omega_D < |\xi_k| < \omega_C, \\ 0 & \text{else.} \end{cases}$$

Find the critical temperature for this model.

Background information: The resulting formula for T_c is a special case of McMillan's formula[1], derived from the linearized Eliashberg equations. In the original form, McMillan's formula has three parameters: the electron phonon coupling strength λ, the Coulomb pseudo-potential μ^* and the average phonon frequency $\langle\omega\rangle$,

$$\frac{T_c}{\omega_D} \propto \exp\left(-\frac{1+\lambda}{\lambda - \lambda\mu^*\frac{\langle\omega\rangle}{\omega_D} - \mu^*}\right).$$

Making the simplifying assumption $\langle\omega\rangle = \omega_D$ and identifying $\lambda = \frac{g\nu}{1-g\nu}$ and $\mu^{*-1} = (V\nu)^{-1} + \ln\frac{\omega_C}{\omega_D}$, where $\nu = \nu(E_F)$ is the density of states at the Fermi level, you should obtain the result from the exercise.

[1] W. L. McMillan, "Transition temperature of strong-coupled superconductors", Phys. Rev. 167 (1968), pp. 331–344. DOI:10.1103/PhysRev.167.331