Institut für Theoretische Festkörperphysik
Karlsruher Institut für Technologie

Prof. Dr. M. Garst
Dr. J. Masell
Dr. S. Sorn

Condensed Matter Theory 1 — Exercise 14

Winter term 2023/24

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To be discussed on: *Thursday 2024/02/15*

1. Josephson effect

The aim of this exercise is to calculate the Josephson current between two superconductors for $T = 0$. The two superconductors are labeled by 1 and 2 and are described by the BCS-Hamiltonians

$$\mathcal{H}^{\mathrm{MF}}_{\mathrm{BCS},1} = \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k},\sigma} c_{\boldsymbol{k},\sigma} - \sum_{\boldsymbol{k}} \Delta_1^* c_{-\boldsymbol{k},\downarrow} c_{\boldsymbol{k},\uparrow} - \sum_{\boldsymbol{k}} \Delta_1 c^\dagger_{\boldsymbol{k},\uparrow} c^\dagger_{-\boldsymbol{k},\downarrow} \tag{1}$$

and

$$\mathcal{H}^{\mathrm{MF}}_{\mathrm{BCS},2} = \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} d^\dagger_{\boldsymbol{k},\sigma} d_{\boldsymbol{k},\sigma} - \sum_{\boldsymbol{k}} \Delta_2^* d_{-\boldsymbol{k},\downarrow} d_{\boldsymbol{k},\uparrow} - \sum_{\boldsymbol{k}} \Delta_2 d^\dagger_{\boldsymbol{k},\uparrow} d^\dagger_{-\boldsymbol{k},\downarrow}, \tag{2}$$

respectively, where $\Delta_{1,2} = |\Delta| e^{i\phi_{1,2}}$. The electron annihilation operators for 1 and 2 are $c_{\boldsymbol{k},\sigma}$ and $d_{\boldsymbol{k},\sigma}$, resepctively, and the electron energy for both superconductors is $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$. The tunneling of electrons between the two superconductors is described by the Hamiltonian

$$\mathcal{H}_{\mathrm{T}} = \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\sigma} V \left(c^\dagger_{\boldsymbol{k}_1,\sigma} d_{\boldsymbol{k}_2,\sigma} + d^\dagger_{\boldsymbol{k}_2,\sigma} c_{\boldsymbol{k}_1,\sigma} \right). \tag{3}$$

Here V is the spin-independent tunneling amplitude. An electron does not change its spin during the tunneling process. The total Hamiltonian reads

$$\mathcal{H} = \mathcal{H}^{\mathrm{MF}}_{\mathrm{BCS},1} + \mathcal{H}^{\mathrm{MF}}_{\mathrm{BCS},2} + \mathcal{H}_{\mathrm{T}}. \tag{4}$$

a) Perform a gauge transformation so that the two order parameters Δ_1 and Δ_2 become real. The phases ϕ_1 and ϕ_2 should then appear in $\mathcal{H}_{\mathrm{T}}$.

b) The current operator in Heisenberg representation is defined as

$$I = -e\dot{N}_1 = -\frac{ie}{\hbar} \left[\mathcal{H}, N_1 \right], \tag{5}$$

where $N_1 = \sum_{\boldsymbol{k},\sigma} c^\dagger_{\boldsymbol{k},\sigma} c_{\boldsymbol{k},\sigma}$ is the particle number operator. Obtain the operator I by evaluating the commutator. The term proportial to V is the tunnel current into the other superconductor which is important in the following. The other term, which is proportional to $|\Delta|$, is the current due to local pair creation/annihilation in the superconductor and can be disregarded for the following tasks.

We cannot exactly compute the expectation value $\langle I(t) \rangle$ of the current through the tunnel barrier since the states are not known. However, in the following, we derive a perturbative result, assuming that $\mathcal{H}_\mathrm{T}$ is a small perturbation that has been switched on adiabatically in the far past ($t \to -\infty$). We thus split the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \delta\mathcal{H}$ into a time-independent and time-dependent contribution, $\mathcal{H}_0 = \mathcal{H}^{\mathrm{MF}}_{\mathrm{BCS},1} + \mathcal{H}^{\mathrm{MF}}_{\mathrm{BCS},2}$ and $\delta\mathcal{H} = \mathcal{H}_\mathrm{T}$.

c) The time evolution of the density matrix $\rho(t)$ is known to be $\partial_t\rho(t) = -\frac{i}{\hbar}[\mathcal{H}(t), \rho(t)]$. For treating $\delta\mathcal{H}$ as a perturbation, introduce the density matrix in the *interaction picture* which is given by

$$\rho_i(t) = e^{i\mathcal{H}_0 t}\rho(t)e^{-i\mathcal{H}_0 t}. \tag{6}$$

Compute $\partial_t\rho_i(t)$ and derive the perturbative expression

$$\rho_i(t) \approx \rho_0 - \frac{i}{\hbar}\int_{-\infty}^{t} [\delta\mathcal{H}_i(t'), \rho_0]dt' + \mathcal{O}(\delta\mathcal{H}_i^2) \tag{7}$$

where $\delta\mathcal{H}_i(t')$ is the perturbation in the interaction picture and ρ_0 the density matrix at $t \to -\infty$.

d) Consider the case $T = 0$, i.e., both superconductors are in the groundstate. Use the first-order time-dependent perturbation theory result from Eq. (6) to bring the expectation value $\langle I(t)\rangle = \mathrm{tr}\{\rho(t)I(t)\}$ into the form of an expectation value with respect to the groundstate of the superconductors.

e) The current between the superconductors is known as *Josephson current*. Compute the Josephson current and show that it takes the form $I = I_c \sin(\phi_1 - \phi_2)$.
Bonus: What is the constant I_c?