

Aufgabe 4a)

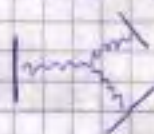
$$V = \frac{1}{3!} \int d\mathbf{r}_1^3 \int d\mathbf{r}_2^3 \int d\mathbf{r}_3^3 \psi^\dagger(1) \psi^\dagger(2) \psi^\dagger(3)$$

$$V = \frac{1}{3!} \int d\mathbf{r}_1^3 \int d\mathbf{r}_2^3 \int d\mathbf{r}_3^3 \psi^\dagger(1) \psi^\dagger(2) \psi^\dagger(3) V(1,2,3) \psi(3) \psi(2) \psi(1)$$

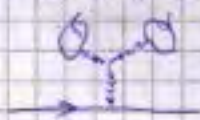
$$1 \equiv \mathbf{r}_1, t_1 \quad V(1,2,3) = V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \delta(t_1 - t_2) \delta(t_2 - t_3)$$

$$= \frac{1}{3!} \sum_{\sigma \in S_3} \int d\mathbf{r}_1^3 \int d\mathbf{r}_2^3 \int d\mathbf{r}_3^3 \psi^\dagger_{\sigma(1)} \psi^\dagger_{\sigma(2)} \psi^\dagger_{\sigma(3)} V(1,2,3) \psi_{\sigma(3)} \psi_{\sigma(2)} \psi_{\sigma(1)}$$

b) Wir zeichnen alle topologisch verschiedene verbundene Diagramme mit 3n Vertizes. $n = \frac{3}{2}$



Bsp.



1) entspricht $G_0(2,1)$

2) entspricht $V(1,2,3)$

3) $\int d\mathbf{r} d\mathbf{r}'$ über alle 3n Koordinaten

4) $(-1)^{n_s}$ für jeden "Fermi" Loop
 n_s für jeden "Fermi" Loop

5) n-te Ordnung

Faktor ~~$\frac{(-1)^n}{n!}$~~ $\frac{(-1)^n}{n!}$, n! kommt auch mit
Vorzeichen des Vertices

$\left(\frac{1}{3!}\right)^n \Rightarrow$ Symmetrie Faktor

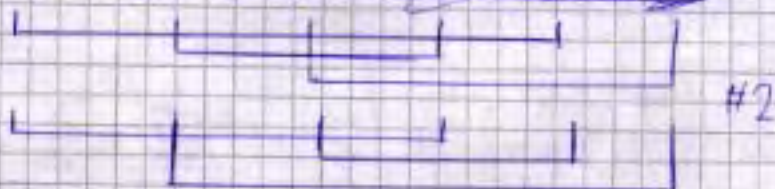
$$\left(\frac{1}{3!}\right)^n \cdot \# \text{ Mögl. Vertices} = \frac{1}{15}$$

Aufgaben Teil (c)

$\psi^+(1) \psi^+(2) \psi^+(3) \psi(3) \psi(2) \psi(1)$



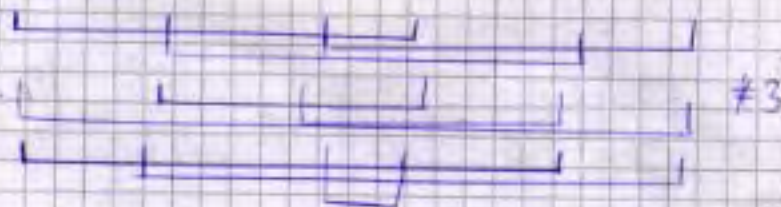
$P=3$



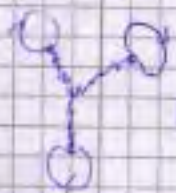
#2



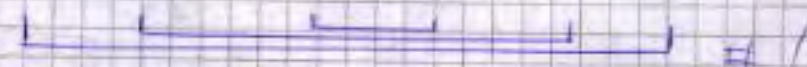
$P=2$



#3



$P=6$



#1

$$\Delta E^{(1)} = \langle V \rangle = \frac{1}{15} \sum \text{Diagramme} \quad (T=\beta)$$

$$\frac{1}{3!} \cdot \# \text{ Mögl. Vertices} = \frac{1}{15}$$

$$\text{Diagram 1} = \frac{1}{3} n_s (-1) (-1) \int d1 \int d2 \int d3 G(2,1) G(3,2) G(1,3) \times V(1,2,3)$$

$$= \frac{1}{3} n_s (-1) (-1) \lambda \beta \int d\mathbf{r} G^3(\mathbf{r}, i; \mathbf{r}, j)$$

$$= \frac{1}{3} \lambda \beta V \rho^3$$

$$\text{Diagram 2} = \frac{1}{2} V \beta \lambda \rho^3 (-1) (-1)^2 n_s^2$$

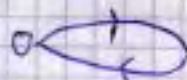
$$\text{Diagram 3} = \frac{1}{6} V \beta \lambda \rho^3 n_s^3 (-1) (-1)^3$$

$$\Rightarrow \Delta E' = V \lambda \rho^3 \left(\frac{n_s}{3} - \frac{n_s^2}{2} + \frac{n_s^3}{6} \right)$$

$$= \frac{1}{6} V \lambda \rho^3 n_s (n_s - 1) (n_s - 2)$$

Aufgabe 2a)

$$\begin{aligned}
 & - \langle T C_k^\dagger(J) C_{k+q}(J) C_{k'}^\dagger(0) C_{k'-q}(0) \rangle \\
 & = - \underbrace{\langle T C_k^\dagger(J) C_{k+q}(J) \rangle}_{0} \langle T C_{k'}^\dagger C_{k'-q}(0) \rangle \\
 & \quad + \langle T C_{k+q}(J) C_{k'}^\dagger(0) \rangle \langle T C_{k'-q}(0) C_k^\dagger(J) \rangle \\
 & = + \delta_{k+q, k'} G_{k+q}(J, 0) \delta_{k'-q, k} G_k(0, J)
 \end{aligned}$$

Diagram: 

$$G_{k+q}(J) G_k(-J) = \frac{1}{\beta^2} \sum_{n'} G_k(i\omega_{n'}) e^{i\omega_{n'} J} \sum_{n''} G_k(i\omega_{n''}) e^{-i\omega_{n''} J}$$

$$\omega_{n''} \rightarrow \omega_{n'} + \omega_n$$

$$= \frac{1}{\beta} \sum_n e^{-i\omega_n J} \underbrace{\frac{1}{\beta} \sum_{n'} G_k(i\omega_{n'}) G_{k+q}(i\omega_{n'} + i\omega_n)}_{\chi_0^R(k, q, i\omega_n)}$$

$$\omega_{n''} = (2n''+1)\frac{\pi}{\beta} = (2n'+1)\frac{\pi}{\beta} + \underbrace{2n\frac{\pi}{\beta}}_{\omega_n}$$

$$\chi_0^R(k, q, i\omega_n) = \frac{n_F(\epsilon_k) - n_F(\epsilon_{k+q})}{i\omega_n - (\epsilon_{k+q} - \epsilon_k)}$$

Aufgabe 2b

$$H = H_0 + H_{\text{int}}$$

$$H_0 = \sum_k \epsilon_k C_k^\dagger C_k \quad H_{\text{int}} = \frac{1}{2V} \sum_{k, k', q \neq 0} V_c(q) C_{k+q}^\dagger C_{k-k+q}^\dagger C_{k'} C_{k'}$$

$$\hat{\rho}_q(z) = \sum_k \rho_{kq}(z) \quad \rho_{kq}(z) = C_k^\dagger C_{k+q}$$

$$\chi^R(k, q, z) = - \langle T_z C_k^\dagger(z) C_{k+q}(z) \hat{\rho}_{-q}(0) \rangle$$

Suche Bewegungsgl: $\frac{d}{dz} \chi^R(k, q, z)$

$$[H_{\text{int}}, C_k^\dagger(z) C_{k+q}(z)] = - \frac{1}{2V} \sum_{k', q' \neq 0} V_c(q') [C_{k+q'}^\dagger C_{k-k'+q'}^\dagger C_{k'} C_{k'} + C_{k+q'}^\dagger C_{k-k'+q'}^\dagger C_{k+q} C_{k'} - C_{k+q'}^\dagger C_k^\dagger C_{k+q+q'} C_{k'} - C_k^\dagger C_{k-k'+q'}^\dagger C_{k'} C_{k+q+q'}]$$

$$\text{Bsp 1: } C_{k+q'}^\dagger C_{k-k'+q'}^\dagger C_{k'} C_{k+q} = C_{k+q'}^\dagger C_{k+q} n_F(\epsilon_{k'}) \delta_{q', 0} + n_F(\epsilon_k) \delta_{q', -q} C_{k-k'+q'}^\dagger C_{k'}$$

Damit ergibt sich

$$[H_{\text{int}}, C_k^\dagger(z) C_{k+q}(z)] = + V_c(q) (n_F(\epsilon_{k+q}) - n_F(\epsilon_k)) \frac{1}{V} \sum_{k'} C_{k-k+q}^\dagger C_{k'}$$

$$\text{Bsp 2: } C_{k+q'}^\dagger C_{k-k'+q'}^\dagger C_{k+q} C_{k'} = C_{k+q'}^\dagger C_{k'} n_F(\epsilon_k) \delta_{q', -q} + C_{k-k'+q'}^\dagger C_{k+q} n_F(\epsilon_{k'}) \delta_{q', 0}$$

Nebenrechnung 2b)

$$[C_{n+q}^+, C_{n'-q}^+, C_{\bar{n}}^+, C_{\bar{n}}^+, C_{\bar{n}}^+, C_{n+q}]$$

$$= C_{n+q}^+ C_{n'-q}^+ C_{\bar{n}}^+ [C_{\bar{n}}^+, C_{\bar{n}}^+ C_{n+q}]$$

$$+ [C_{n+q}^+, C_{n'-q}^+ C_{\bar{n}}^+, C_{\bar{n}}^+ C_{n+q}] C_{\bar{n}}^+ = *$$

$$[C_{\bar{n}}^+, C_{\bar{n}}^+ C_{n+q}]$$

$$= C_{\bar{n}}^+ C_{\bar{n}}^+ C_{n+q} - C_{\bar{n}}^+ C_{n+q} C_{\bar{n}}^+$$

$$= \delta_{\bar{n}k} C_{n+q} - C_{\bar{n}}^+ C_{\bar{n}}^+ C_{n+q} - C_{\bar{n}}^+ C_{n+q} C_{\bar{n}}^+$$

$$= \delta_{\bar{n}k} C_{n+q}$$

~~110 $[C_{n+q}^+, C_{n'-q}^+ C_{\bar{n}}^+, C_{\bar{n}}^+ C_{n+q}] C_{\bar{n}}^+ =$~~

~~$[C_{n+q}^+, C_{n'-q}^+ C_{\bar{n}}^+, C_{\bar{n}}^+ C_{n+q}] C_{\bar{n}}^+ =$~~

$$[C_{n+q}^+, C_{n'-q}^+ C_{\bar{n}}^+] =$$

$$C_{n+q}^+ C_{n'-q}^+ C_{\bar{n}}^+ - C_{n'-q}^+ C_{\bar{n}}^+ C_{n+q}^+$$

$$= -\delta_{n'-q, k+q} C_{\bar{n}}^+$$

$$* = + C_{n+q}^+ C_{n'-q}^+ C_{\bar{n}}^+ \delta_{\bar{n}k} C_{n+q}$$

$$+ C_{n+q}^+ C_{n'-q}^+ [C_{\bar{n}}^+, C_{\bar{n}}^+ C_{n+q}] C_{\bar{n}}^+$$

$$+ [C_{n+q}^+, C_{n'-q}^+ C_{\bar{n}}^+, C_{\bar{n}}^+ C_{n+q}] C_{\bar{n}}^+ C_{\bar{n}}^+$$

$$= + C_{n+q}^+ C_{n'-q}^+ C_{\bar{n}}^+ \delta_{\bar{n}k} C_{n+q}$$

$$+ C_{n+q}^+ C_{n'-q}^+ \delta_{\bar{n}k} C_{n+q} C_{\bar{n}}^+$$

$$+ C_{n+q}^+ [C_{n'-q}^+, C_{\bar{n}}^+ C_{n+q}] C_{\bar{n}}^+ C_{\bar{n}}^+$$

$$+ [C_{n+q}^+, C_{\bar{n}}^+ C_{n+q}] C_{n'-q}^+ C_{\bar{n}}^+ C_{\bar{n}}^+$$

$$= C_{n+q}^+ C_{n'-q}^+ C_{\bar{n}}^+ \delta_{\bar{n}k} C_{n+q}$$

$$+ C_{n+q}^+ C_{n'-q}^+ \delta_{\bar{n}k} C_{n+q} C_{\bar{n}}^+ - C_{n+q}^+ \delta_{n'-q, k+q} C_{\bar{n}}^+ C_{\bar{n}}^+ C_{\bar{n}}^+$$

$$+ - \delta_{n'-q, k+q} C_{\bar{n}}^+ C_{n'-q}^+ C_{\bar{n}}^+ C_{\bar{n}}^+$$

Erinnerung:

$$[H_0, c_n^\dagger(\beta) c_{n+q}(\beta)] = (\epsilon_n - \epsilon_{n+q}) c_n^\dagger c_{n+q}$$

$$\begin{aligned} \langle [c_n^\dagger c_{n+q}, \beta - \alpha] \rangle &= \langle c_n^\dagger c_n \rangle - \langle c_{n+q}^\dagger c_{n+q} \rangle \\ &= n_F(\epsilon_n) - n_F(\epsilon_{n+q}) \end{aligned}$$

$$\begin{aligned} -\frac{\partial}{\partial \beta} \chi^R(k, q, \beta) &= + (n_F(\epsilon_n) - n_F(\epsilon_{n+q})) \beta(\beta) \\ &\quad - (\epsilon_n - \epsilon_{n+q}) \chi^R(k, q, \beta) \\ &\quad + (n_F(\epsilon_n) - n_F(\epsilon_{n+q})) V_c(q) \\ &\quad \frac{1}{V} \sum_{k'} \chi^R(k', q, \beta) \end{aligned}$$

$$\begin{aligned} [i\omega_n - (\epsilon_{n+q} - \epsilon_n)] \chi^R(k, q, \beta) &= [n_F(\epsilon_n) - n_F(\epsilon_{n+q})] \\ &\quad + (n_F(\epsilon_n) - n_F(\epsilon_{n+q})) V_c(q) \frac{1}{V} \sum_{k'} \chi^R(k', q, \beta) \end{aligned}$$

$$\chi^R(k, q, i\omega_n) = \chi_0^R(k, q, i\omega_n) + \chi_0(k, q, i\omega_n) V_c \chi^R(q, i\omega_n)$$

$$\Rightarrow \chi^R(q, i\omega_n) = \chi_0^R(q, i\omega_n) + \chi_0(q, i\omega_n) V_c(q) \chi^R(q, i\omega_n)$$

$$\Rightarrow \chi^R(q, i\omega_n) = \frac{\chi_0^R(q, i\omega_n)}{1 - \chi_0(q, i\omega_n) V_c(q)}$$

Aufgabe 2 c

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \dots$$

Diagram 1: A circle with a vertical line through its center, shaded on the left side.

Diagram 2: A circle with a vertical line through its center, shaded on the left side.

Diagram 3: A circle with a vertical line through its center, shaded on the left side.

Diagram 4: A circle with a vertical line through its center, shaded on the left side.

$$\approx \text{Diagram 5} + \text{Diagram 6} + \left[\text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \dots \right]$$

Diagram 5: A circle with a vertical line through its center, shaded on the left side.

Diagram 6: A circle with a vertical line through its center, shaded on the left side.

Diagram 7: A circle with a vertical line through its center, shaded on the left side.

Diagram 8: A circle with a vertical line through its center, shaded on the left side.

Diagram 9: A circle with a vertical line through its center, shaded on the left side.

Diagram 10: A circle with a vertical line through its center, shaded on the left side.

$$= \text{Diagram 11} + \text{Diagram 12} + \text{Diagram 13}$$

Diagram 11: A circle with a vertical line through its center, shaded on the left side.

Diagram 12: A circle with a vertical line through its center, shaded on the left side.

Diagram 13: A circle with a vertical line through its center, shaded on the left side.

$$\chi^R(q, i\omega) = \chi_0^R(q, i\omega) + \chi_0^R(q, i\omega) V(q) \chi^R(q, i\omega)$$

Aufgabe 3 a)

Das abgeschirmte Coulombpotential

$$\begin{aligned} \Rightarrow \Rightarrow &= \dots + \dots + \dots + \dots + \dots \\ &= \dots + \dots + \dots + \dots + \dots \\ &= \dots + \dots + \dots + \dots + \dots \end{aligned}$$

$$\tilde{V}(q) = V(q) + V(q) \chi_0^R(q, i\omega_n) \tilde{V}(q, i\omega_n)$$

$$\Rightarrow \tilde{V}_{\text{eff}} = \frac{1}{1 - V_c(q, \omega) \chi_0^R(q, \omega)} V_c(q, \omega)$$

$$1 + V_c \chi^R = \frac{1}{1 - V_c \chi_0^R} \Rightarrow V_c \chi^R = \frac{V_c \chi_0^R}{1 - V_c \chi_0^R}$$

$$\chi^R = \frac{\chi_0^R}{1 - V_c \chi_0^R}$$

$$\chi_0^R(q, \omega) = -v(0) \text{ for } \omega = 0$$

$$V_c(r-r') = \frac{1}{4\pi\epsilon_0} \frac{1}{|r-r'|}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r}|} = \lim_{k \rightarrow 0} \frac{1}{4\pi\epsilon_0} \frac{e^{-k|\vec{r}|}}{|\vec{r}|}$$

$$\int d^3r \frac{e^{-k|\vec{r}|}}{|\vec{r}|} e^{i\vec{k} \cdot \vec{r}} \quad // \int$$

wähle: $\vec{k} \cdot \vec{r} = k_z r_z = |\vec{k}| r \cos \vartheta$

$$= \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos \vartheta \int_0^\infty dr r^2 \frac{e^{-kr}}{r} e^{i|\vec{k}|r \cos \vartheta}$$

$$= 4\pi \int_0^\infty dr \frac{e^{-kr}}{|\vec{k}|} \sin(kr)$$

$$= 4\pi \frac{1}{|\vec{k}|^2 + k^2}$$

$$\Rightarrow V(q) = \frac{1}{\epsilon_0} \frac{1}{|q|^2}$$

$$\tilde{V}_c = \frac{V_c(q)}{1 + v(q)V_c(q)} = \frac{1}{\epsilon_0(|q|^2 + k^2)} \quad k^2 = \sqrt{\epsilon_0' V_c(0)}$$

$$\Rightarrow \tilde{V}_c(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{e^{-k|\vec{r}|}}{|\vec{r}|}$$

Aufgabe 3b

$$\chi_0^R(q, \omega) = \int \frac{d^3k}{(2\pi)^3} \frac{n_F(k) - n_F(k+q)}{\omega - (E_{k+q} - E_k)}$$

$$E_k = \epsilon_k - \mu$$

$$\approx \int \frac{d^3k}{(2\pi)^3} \frac{\vec{q} \cdot \vec{v}_k}{\omega - \vec{q} \cdot \vec{v}_k} \left(- \frac{dn_F}{d\epsilon} \right)$$

$$v_k = \vec{v}_k = \frac{1}{\hbar} \nabla_k \epsilon_k$$

$$\approx - \int \frac{d^3k}{(2\pi)^3} \frac{(\vec{q} \cdot \vec{v}_k)^2}{\omega^2} \frac{dn_F}{d\epsilon}$$

$$= - \frac{|q|^2}{\omega^2} \int \frac{d^3k}{(2\pi)^3} \frac{|v_k|^2 \cos^2 \theta}{\omega^2} \frac{dn_F}{d\epsilon}$$

$$= - \frac{|q|^2}{\omega^2} \frac{1}{4\pi^2} \int_{-1}^1 d\cos \theta \int_0^\infty dk k^2 |v_k|^2 \frac{dn_F}{d\epsilon} \cos^2 \theta$$

$$= - \frac{|q|^2}{\omega^2} \frac{1}{8\pi^2} \int_0^\infty dk k^2 |v_k|^2 \frac{dn_F}{d\epsilon}$$

$$\frac{dn_F}{d\epsilon} = -\delta(\epsilon - \epsilon_F)$$

$$dk = d\epsilon \frac{1}{\left| \frac{d\epsilon}{dk} \right|} = \frac{d\epsilon}{|v_k|^2}$$

$$|v_k| = \frac{k_F}{m} \quad \bar{n} = \frac{k_F^3}{2\pi^2}$$

$$\Rightarrow \chi_0^R = \frac{|q|^2}{\omega^2} \frac{n}{m}$$

$$\epsilon^+ = \frac{1}{1 - V_C(q) \chi_0^R(q)} = \frac{1}{1 - \frac{\hbar^2}{m \epsilon_0} \frac{1}{\omega^2}}$$

$$\omega_p = \sqrt{\frac{n}{m \epsilon_0}}$$