Theorie der Kondensierten Materie II   SS 2015

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1. Jordan-Wigner Transformation:

(a) Consider the set of Pauli matrices σ_n^α satisfying the usual SU(2) commutation relations for each n, but commuting for different n.
Show that the following transformation:

$$\sigma_n^z = 2a_n^\dagger a_n - 1, \quad \sigma_n^- = a_n \prod_{m<n} \sigma_m^z, \quad \sigma_n^+ = a_n^\dagger \prod_{m<n} \sigma_m^z,$$

maps the above set of Pauli matrices into a set of fermionic operators.
Show that the operators a_n defined by the transformation obey the fermionic commutation relations for each n and anticommute for different n.

(b) Consider the one-dimensional spin chain, described by the generic Hamiltonian

$$\hat{H} = \sum_{n=-\infty}^{\infty} \left(J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z - B\sigma_n^z \right),$$

where σ_n^α are the Pauli matrices.
Use the above Jordan-Wigner transformation to express $\hat{H}$ in terms of fermions.

2. Bogolyubov transformation:

Consider now the simper case of the "quantum Ising model", described by the above Hamiltonian $\hat{H}$ with $J_z = J_y = 0$. Observe that this Hamiltonian is quadratic in fermionic operators.

(a) Find a unitary transformation diagonalizing the quantum Ising model.

(b) Discuss the spectrum of the model. Find the dependence of the spectral gap on the applied field. Is there a point, where the spectrum is gapless (the so-called "quantum critical point")?

(c) What changes in your analysis for $J_y \neq 0$?