Theorie der Kondensierten Materie II SS 2015

Prof. Dr. A. Shnirman
Dr. B. Narozhny

Blatt 1
Lösungen

1. Scattering Amplitude:

The scattering state

$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\vec{r}} + \chi_{\vec{k}}(\vec{r}), \quad \chi_{\vec{k}}(\vec{r}) = f(\vec{k}, k\vec{n})\frac{e^{ik|\vec{r}|}}{|\vec{r}|},$$

can be found from the Green's function formalism as follows.

First, one can write down the Schrödinger equation describing a particle moving in a given potential $V(\vec{r})$ in the momentum representation with the help of the single-particle Green's function:

$$\left[\hat{G}_0^{-1} - \hat{V}\right]\psi_{\vec{k}}(\vec{r}) = 0.$$

Recall, that in the momentum representation the free-particle Green's function is

$$\hat{G}_0 = \frac{1}{\epsilon - \vec{p}^{\,2}/(2m) + i\delta},$$

while the potential $\hat{V}$ is actually an (integral) operator.

Substituting the above scattering wave function and noticing that the plane wave is a solution of the Schrödinger equation for a free particle, one finds

$$\left[\hat{G}_0^{-1} - \hat{V}\right]\chi_{\vec{k}}(\vec{r}) = \hat{V}e^{i\vec{k}\vec{r}} = \hat{V}|\vec{k}\rangle.$$

The solution can be formally written don using the "full" Green's function

$$\hat{G}^{-1} = \hat{G}_0^{-1} - \hat{V} \quad \Rightarrow \quad \chi_{\vec{k}}(\vec{r}) = \langle\vec{r}|\hat{G}\hat{V}|\vec{k}\rangle.$$

Now, we can expand the Green's function into a power series

$$\hat{G} = \hat{G}_0 + \hat{G}_0\hat{V}\hat{G}_0 + \dots$$

This allows us to write the solution for $\chi_{\vec{k}}(\vec{r})$ as

$$\chi_{\vec{k}}(\vec{r}) = \langle\vec{r}|\hat{G}_0\hat{V} + \hat{G}_0\hat{V}\hat{G}_0\hat{V} + \dots|\vec{k}\rangle = \langle\vec{r}|\hat{G}_0\hat{F}|\vec{k}\rangle,$$

where

$$\hat{F} = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \dots$$

This series can be pictorially represented by the diagrams shown in the original figure.

To relate the quantity $\hat{F}$ to the scattering amplitude, consider the following expression for the free-particle Green's function ($\epsilon = k^2/(2m)$)

$$G_0(\epsilon; \vec{r}, \vec{r}\,') = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{r}-\vec{r}\,')}}{\epsilon - \vec{p}^{\,2}/(2m) + i\delta} = -\frac{m}{2\pi} \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}.$$

Now we are going to consider this Green's functions "at large distances". The meaning of this phrase is the following. We choose the coordinate system in such a way that the scattering center is located near the origin, while the vector $\vec{r}$ points towards the observation point. The vector $\vec{r}\,'$ spans the area around the origin, where the potential $V(\vec{r}\,')$ is nonzero (in a scattering problem we are looking at a potential that is confined to a certain area and study how particles – or waves – arriving from infinity scatter off this potential). Now we denote the length of the vector $\vec{r}$ by R and assume all other lengths in the problem to be much smaller:

$$\vec{r} = R\vec{n}, \quad |\vec{r}-\vec{r}\,'| \approx R - |\vec{r}\,'|\cos\theta + \mathcal{O}(1/R), \quad \cos\theta = \frac{\vec{n}\cdot\vec{r}\,'}{r'},$$

where θ is the angle between $\vec{n}$ and $\vec{r}\,'$.

Substituting the above approximation into the single-particle Green's function, we find

$$\chi_{\vec{k}}(\vec{r}) = -\frac{me^{ikR}}{2\pi R} \int d^3r' e^{-ik|\vec{r}\,'|\cos\theta} \langle \vec{r}\,'|\hat{F}|\vec{k}\rangle.$$

Comparing this expression with the definition of the scattering amplitude, we find the relation

$$f(\vec{k}_1, \vec{k}_2) = -\frac{m}{2\pi} \langle \vec{k}_2|\hat{F}|\vec{k}_1\rangle,$$

where

$$\vec{k}_2 = \left|\vec{k}_1\right| \vec{n}.$$

Let us now derive the integral equation for the scattering amplitude. We re-write the series expantion for $\hat{F}$ in the momentum representation:

$$\hat{F} = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \ldots \quad \Rightarrow \quad F(\vec{k}_1, \vec{k}_2) = F^{(1)}(\vec{k}_1, \vec{k}_2) + F^{(2)}(\vec{k}_1, \vec{k}_2) + \ldots$$

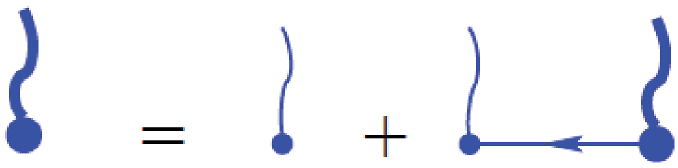
$$F^{(1)}(\vec{k}_1, \vec{k}_2) = V(\vec{k}_2 - \vec{k}_1),$$

$$F^{(2)}(\vec{k}_1, \vec{k}_2) = \int d^3q \frac{V(\vec{k}_2 - \vec{q})V(\vec{q} - \vec{k}_1)}{\epsilon - q^2/(2m) + i\delta}.$$

Now we can see, that in the series of diagrams in the figure the straight lines correspond to free-particle Green's functions and the curvy lines – to the matrix elements of the scattering potential. All internal momenta should be integrated over, while the incoming and outgoing momenta should be "on shell", i.e. should satisfy $\epsilon = k^2/(2m)$.

The integral equation for $\hat{F}$ can be easily expressed either diagrammatically, or in the operator form:

$$\hat{F} = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \hat{V}\hat{G}_0\hat{V}\hat{G}_0\hat{V} + \cdots = \hat{V} + \hat{V}\hat{G}_0(\hat{V} + \hat{V}\hat{G}_0\hat{V} + \ldots) = \hat{V} + \hat{V}\hat{G}_0\hat{F}.$$



In the momentum representation the integral equation has the form

$$F(\vec{k}_1, \vec{k}_2) = V(\vec{k}_2 - \vec{k}_1) + 2m \int \frac{d^3q}{(2\pi)^3} \frac{V(\vec{k}_2 - \vec{q}) F(\vec{k}_1, \vec{q})}{\epsilon - q^2/(2m) + i\delta}.$$

2. Second quantization:

See the attached problems and their solutions.

$$U_{\pm}(x) = W_0^2 \mp \alpha\delta(x), \quad \alpha = \sqrt{\frac{2}{m}}\hbar W_0,$$

and the existence of the zero-energy state is connected to the presence of a single discrete level in a δ-well; see Problem 2.7.

10.3 The simplest systems with a large number of particles ($N \gg 1$)

Problem 10.28

For the ground state of a *Bose*-gas consisting of N identical non-interacting particles with the spin $s = 0$ in a volume V, calculate the mean particle number density, the mean number of particles in a volume v, and the fluctuation of this particle number.

Solve this problem by averaging the physical operators in the occupation-number representation.

Solution

Expressing $\psi(\mathbf{r})$-operators in terms of creators $\hat{a}^+_{\mathbf{k}}$ and annihilators $\hat{a}_{\mathbf{k}}$ of a particle with a given momentum $\mathbf{p} = \hbar\mathbf{k}$, the operator for particle number density $\hat{n}(\mathbf{r}) = \hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r})$ (see Problem 10.22) (here $\hat{\psi}(\mathbf{r}) = \sum\limits_{\mathbf{k}}(1/\sqrt{V})e^{i\mathbf{k}\cdot\mathbf{r}}\hat{a}_{\mathbf{k}}$) becomes

$$\hat{n}(r) = \frac{1}{V}\sum_{\mathbf{k}_1\mathbf{k}_2} e^{i(\mathbf{k}_2-\mathbf{k}_1)r}\hat{a}^+_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}. \tag{1}$$

The mean particle number density $\overline{n(\mathbf{r})}$ is obtained from operator (1) by averaging over the ground state $|\psi_0\rangle = |N_{k=0}0_{k\neq 0}\rangle$ (all the particles have zero momentum $p = 0$). Since

$$\langle\psi_0|\hat{a}^+_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}|\psi_0\rangle = \begin{cases} N, & k_1 = k_2 = 0, \\ 0, & \text{in all other cases,} \end{cases}$$

we have the natural result $\overline{n} = N/V$.

The mean particle number in the volume v is obtained by averaging the operator $\hat{N}(v) = \int\limits_v \hat{n}(\mathbf{r})d^3r$, and is equal to $\overline{N(v)} = Nv/V$.

To calculate the fluctuations of particle number, we first average the operator $\overline{N^2(v)}$ over the state $|\psi_0\rangle$. Since

$$N^2(v) = \frac{1}{V^2}\int\limits_v\int\limits_v \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \exp\{i[(\mathbf{k}_2 - \mathbf{k}_1)\cdot\mathbf{r} + (\mathbf{k}_4 - \mathbf{k}_3)\cdot\mathbf{r}']\}\hat{a}^+_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}\hat{a}^+_{\mathbf{k}_3}\hat{a}_{\mathbf{k}_4}\ d^3rd^3r',$$

we must first find the matrix elements

$$\langle\psi_0|\hat{a}^+_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}\hat{a}^+_{\mathbf{k}_3}\hat{a}_{\mathbf{k}_4}|\psi_0\rangle$$

to calculate $\overline{N^2(v)}$. Using the explicit form of $|\psi_0\rangle$, we see that the matrix elements are non-vanishing only when conditions $k_1 = k_4 = 0$, $\mathbf{k}_2 = \mathbf{k}_3$ are fulfilled. These elements are equal to N^2 for $\mathbf{k}_2 = \mathbf{k}_3 = 0$ and to N for $\mathbf{k}_2 = \mathbf{k}_3 \equiv \mathbf{k} \neq 0$. Taking this into account we obtain

$$\overline{N^2(v)} = \frac{1}{V^2}\int_v\int_v \left\{ N^2 + N\sum_{k\neq 0} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \right\} d^3r\, d^3r'. \tag{2}$$

Because the functions $\psi_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$ form a complete set, the sum here is equal to $V\delta(\mathbf{r}-\mathbf{r}') - 1$, and integration gives

$$\overline{N^2(v)} = \left(\frac{Nv}{V}\right)^2 + \frac{Nv}{V} - \frac{Nv^2}{V^2}. \tag{3}$$

Hence

$$\overline{(\Delta N(v))^2} = \overline{N^2(v)} - \overline{N(v)}^2 = \frac{Nv}{V}\left(1 - \frac{v}{V}\right). \tag{4}$$

For $v = V$ we have $\overline{(\Delta N(v))^2} = 0$, which is obvious, since the total number of particle in the system is equal to N and does not fluctuate. When $v \ll V$, according to (4), we have

$$\overline{(\Delta N(v))^2} \approx \frac{Nv}{V} = \overline{N(v)}.$$

Let us note that for a system of N non-interacting classical particles in a volume V, the distribution of the number of particles N_v in the volume v has the form

$$W(N_v) = \frac{N!}{N_v!(N-N_v)!}\left(\frac{v}{V}\right)^{N_v}\left(1 - \frac{v}{V}\right)^{N-N_v}$$

(binomial distribution). For such a distribution, the calculation of mean values $\overline{N_v}$, $\overline{N_v^2}$, $\overline{(\Delta N_v)^2}$ gives results that coincide with those obtained above (see also a remark in the next problem).

Problem 10.29

Under the same conditions as in the previous problem, consider the spatial correlation of density fluctuations. For a homogeneous system it is characterized by a *correlation function* $\nu(\mathbf{r})$ $(\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2)$ equal to

$$\nu(\mathbf{r}) = \frac{\overline{n_1 n_2} - \overline{n}^2}{\overline{n}}, \quad n_{1,2} \equiv n(\mathbf{r}_{1,2}),$$

where $\overline{n}$ is the mean particle number density.

Compare this result to the corresponding result for a system of classical particles.

Solution

Since particle density operators in different points of space commute with each other, the operator of the form $\hat{n}_1\hat{n}_2$ can be written:

$$\hat{n}_1\hat{n}_2 = \hat{\psi}^+(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_1)\hat{\psi}^+(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_2)$$
$$= \frac{1}{V^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \exp\{i[(\mathbf{k}_2-\mathbf{k}_1)\mathbf{r}_1 + (\mathbf{k}_4-\mathbf{k}_3)\mathbf{r}_2]\}\hat{a}^+_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}\hat{a}^+_{\mathbf{k}_3}\hat{a}_{\mathbf{k}_4},$$

and its mean value in the Bose-gas ground state is

$$\overline{n_1n_2} = \frac{1}{V^2}\left\{N^2 + N\sum_{k\neq 0} e^{i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)}\right\} = \frac{N}{V}\delta(\mathbf{r}_1-\mathbf{r}_2) + \frac{N^2}{V^2} - \frac{N}{V^2}. \tag{1}$$

Compare this to the derivation of Eqs. (2) and (3) in the previous problem. Hence[223]

$$\frac{1}{\overline{n}}\{\overline{n_1n_2} - \overline{n}^2\} = \delta(r) - \frac{\overline{n}}{N},$$

and the correlation function becomes equal to

$$\nu = -\frac{\overline{n}}{N}. \tag{2}$$

In order to understand the results obtained, we derive correlators similar to (1), (2) for the case of non-interacting classical particles. Taking into account the fact that the probability distribution of particle coordinates is described in terms of a product d^3r_a/V and $n(\mathbf{r}) = \sum\limits_a \delta(\mathbf{r}-\mathbf{r}_a)$, we find

$$\overline{n(\mathbf{r}'_1)n(\mathbf{r}'_2)} = \int\dots\int\sum_{a,b}\delta(\mathbf{r}'_1-\mathbf{r}_a)\delta(\mathbf{r}'_2-\mathbf{r}_b)\frac{d^3r_1}{V}\dots\frac{d^3r_N}{V}$$
$$= \frac{N}{V}\delta(\mathbf{r}'_1-\mathbf{r}'_2) + \frac{N(N-1)}{V^2}, \tag{3}$$

which coincides with Eq. (1). (Note that the term with a δ-function in (3) corresponds to the terms with $a=b$. The number of such terms is N. The second term corresponds to the terms with $a\neq b$, and their number is $N(N-1)$.)

For macroscopic systems, the value of N is extremely large, and therefore the last term in (1) can be omitted and we have $\nu = 0$ (there is no correlation) in (2). On the other hand, for finite values of N we have $\nu \neq 0$. Here the characteristic properties of ν – its independence from $\mathbf{r}$ and its sign $\nu < 0$ – have an intuitive explanation for

[223] The term with the δ-function, that goes to zero as $r\neq 0$, has a universal character and does not depend on the form of the distribution function.

classical particles. Indeed, the value of $\overline{n_1 n_2}$ is lower than $\overline{n}^2$, since a single particle cannot contribute to the particle number density at different points of space at the same time, regardless of the distance between them (in the case where $N \gg 1$ the density at different points in space is determined mainly by the contribution of different particles).

Many characteristics of a Bose-gas in the ground state, considered here and in the previous problem, are the same as for a gas of classical particles. It is not accidental. Indeed, the wavefunction of the ground state for the Bose gas has the form

$$\Psi_0 = \psi_0(\mathbf{r}_1)\psi_0(\mathbf{r}_2)\dots\psi_0(\mathbf{r}_N), \quad \psi_0(r) = \frac{1}{\sqrt{V}},$$

i.e., is a product of single-particle wavefunctions. This is similar to a gas of distinguishable particles. The particles therefore do not interfere with each other, and for each one of them $|\psi_0|^2 = 1/V$, which corresponds to a constant probability distribution over the volume.

Problem 10.30

In the ground state of an ideal *Fermi*-gas of N particles in volume V, find the mean particle number density and the mean particle number in some volume v.

This problem should be solved by averaging the physical operators in the occupation-number representation.

Solution

The particle density operator $\hat{n}(\mathbf{r})$ has the form

$$\hat{n}(\mathbf{r}) = \sum_{\sigma} \hat{n}(\mathbf{r}, \sigma) = \frac{1}{V} \sum_{\sigma} \sum_{\mathbf{k}_1 \mathbf{k}_2} e^{i(\mathbf{k}_2 - \mathbf{k}_1)\mathbf{r}} \hat{a}^+_{\mathbf{k}_1\sigma} \hat{a}_{\mathbf{k}_2\sigma}. \tag{1}$$

Compare to Problem 10.28 and 10.22, $\sigma \equiv s_z$. The ground state of the Fermi-gas is determined by occupation numbers $n_{\mathbf{k}\sigma}$, equal to 1 for $|\mathbf{k}| < k_F$ and 0 for $|\mathbf{k}| > k_F$, so that

$$|\psi_0\rangle = \prod \hat{a}^+_{\mathbf{k}\sigma}|0\rangle, \tag{2}$$

where the product includes operators $\hat{a}^+_{\mathbf{k}\sigma}$ with quantum numbers $(\mathbf{k}\sigma)$ of occupied states. Here the Fermi momentum, $p_F = \hbar k_F$, is found from the condition

$$\sum_{\sigma} \sum_{\mathbf{k}(k<k_F)} 1 = (2s+1) \int_{k<k_F} \frac{V\, d^3k}{(2\pi)^3} = \frac{(2s+1)V k_F^3}{6\pi^2} = N,$$

i.e., we have

$$k_F = \left[\frac{6\pi^2 N}{(2s+1)V}\right]^{1/3}.$$

Since we see that the matrix element $\langle\psi_0|\hat{a}^+_{\mathbf{k}_1\sigma}\hat{a}_{\mathbf{k}_2\sigma}|\psi_0\rangle$ is different from zero (and equal to 1) only for $\mathbf{k}_1 = \mathbf{k}_2 \equiv \mathbf{k}$, and $|\mathbf{k}| \le k_F$, from Eqs. (1) and (2) we obtain

$$\overline{n} = \langle\psi_0|\hat{n}(\mathbf{r})|\psi_0\rangle = \frac{N}{V}$$

(which is expected), and $\overline{n}(\sigma) = \overline{n}/(2s+1)$, while $\overline{N(v)} = \overline{n}v = Nv/V$.

Problem 10.31

Under the conditions of the previous problem, consider the correlation of particle number densities with definite values of spin z-projection at different points in space: find $\overline{n(\mathbf{r}_1, s_{z1})n(\mathbf{r}_2, s_{z2})}$, and compare to the product $\overline{n(\mathbf{r}_1, s_{z1})} \cdot \overline{n(\mathbf{r}_2, s_{z2})}$. Consider the cases of different and identical values of s_{z1} and s_{z2}.

Find the density-density correlation function (see Problem 10.29).

Solution

The operator $n(\mathbf{r}_1, \sigma_1)n(\mathbf{r}_2, \sigma_2)$ has the form

$$\hat{n}(\xi_1)\hat{n}(\xi_2) = \frac{1}{V}\sum_{\{\mathbf{k}\}} \exp\{i[(\mathbf{k}_2 - \mathbf{k}_1)\mathbf{r}_1 + (\mathbf{k}_4 - \mathbf{k}_3)\mathbf{r}_2]\}\hat{a}^+_{\mathbf{k}_1\sigma_1}\hat{a}_{\mathbf{k}_2\sigma_1}\hat{a}^+_{\mathbf{k}_3\sigma_2}\hat{a}_{\mathbf{k}_4\sigma_2}. \quad (1)$$

(Compare to Problems 10.29 and 10.30). Letting $|\psi_0\rangle$ be the wavefunction of the Fermi-gas ground state given in the previous problem, it is easy to see that the matrix element obtained by averaging

$$\langle\psi_0|\hat{a}^+_{\mathbf{k}_1\sigma_1}\hat{a}_{\mathbf{k}_2\sigma_1}\hat{a}^+_{\mathbf{k}_3\sigma_2}\hat{a}_{\mathbf{k}_4\sigma_2}|\psi_0\rangle,$$

for the case $\sigma_1 \neq \sigma_2$, is different from zero (and equal to 1) only for $\mathbf{k}_1 = \mathbf{k}_2$, $\mathbf{k}_3 = \mathbf{k}_4$ and $|\mathbf{k}_{1,3}| \le k_F$. Taking this into account, for $\sigma_1 \neq \sigma_2$ we find

$$\langle\psi_0|\hat{n}(\xi_1)\hat{n}(\xi_2)|\psi_0\rangle = \frac{1}{V^2}\sum_{|\mathbf{k}_{1,2}|\le k_F} 1 = \frac{\overline{n}^2}{(2s+1)^2}, \quad \overline{n} = \frac{N}{V}. \quad (3)$$

(For the calculation of the sum over $\mathbf{k}_{1,2}$, see the previous problem.) Since $\overline{n}(\sigma) = \overline{n}/(2s+1)$, the result (3) means that $\overline{n_1 n_2} = \overline{n_1} \cdot \overline{n_2}$, *i.e*, in the case of different values of the spin projections $\sigma_1 \neq \sigma_2$ there is no correlation between the particle densities at different points in space.

In the case of $\sigma_1 = \sigma_2$, the situation is different. Now the matrix element (2) is different from zero and equal to 1 in the following cases:

1) $\mathbf{k}_1 = \mathbf{k}_2, \quad |\mathbf{k}_2| \le k_F, \quad \mathbf{k}_3 = \mathbf{k}_4, \quad |\mathbf{k}_4| \le k_F;$
2) $\mathbf{k}_1 = \mathbf{k}_4, \quad |\mathbf{k}_4| \le k_F, \quad \mathbf{k}_2 = \mathbf{k}_3, \quad |\mathbf{k}_3| > k_F.$

Using this fact, we find

$$\overline{n(\mathbf{r}_1, \sigma) n(\mathbf{r}_2, \sigma)} = \frac{1}{V^2} \left\{ \sum_{|\mathbf{k}_{1,2}| \le k_F} 1 + \sum_{|\mathbf{k}_1| \le k_F} \sum_{|\mathbf{k}_2| \ge k_F} e^{i(\mathbf{k}_2 - \mathbf{k}_1)(\mathbf{r}_1 - \mathbf{r}_2)} \right\}. \tag{4}$$

Then, using the relation

$$\frac{1}{V} \sum_{|\mathbf{k}| \ge k_F} e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{V} \left\{ \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} - \sum_{|\mathbf{k}| < k_F} e^{i\mathbf{k}\cdot\mathbf{r}} \right\} = \delta(\mathbf{r}) - \frac{1}{V} \sum_{|\mathbf{k}| < k_F} e^{i\mathbf{k}\cdot\mathbf{r}}$$

and calculating the integral (in spherical coordinates with the polar axis directed along the vector $\mathbf{r}$) we obtain:

$$\frac{1}{V} \sum_{|\mathbf{k}| \le k_F} e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{V}{(2\pi)^3} \int_{k \le k_F} e^{i\mathbf{k}\cdot\mathbf{r}} \, d^3k = \frac{V}{2\pi^2 r^2} \left\{ \frac{\sin k_F r}{r} - k_F \cos k_F r \right\},$$

We transform (4) using ($\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\overline{n}(\sigma) = \overline{n}/(2s+1)$):

$$\overline{n(\mathbf{r}_1, \sigma) n(\mathbf{r}_2, \sigma)} = \overline{n(\sigma)}^2 - \frac{1}{4\pi^4 r^4} \left\{ \frac{\sin k_F r}{r} - k_F \cos k_F r \right\}^2.$$

Hence we obtain the correlation function:

$$\nu(\mathbf{r}, \sigma) = -\frac{[\sin k_F r - k_F r \cos k_F r]^2}{4\pi^4 \overline{n}(\sigma) r^6}. \tag{5}$$

Let us discuss this result. The character of the particle density correlations can be made physically clear. Identical particles with different values of spin projection behave like distinguishable ones, so there is no correlation between them. The sign of the correlation function $\nu(\mathbf{r}, \sigma) < 0$ in the case of the same spin projections is also natural. It shows the "repulsive" character of the fermion exchange interaction. For the values $\mathbf{r} = |\mathbf{r}_1 - \mathbf{r}_2| \to \infty$, the correlation disappears.

In conclusion, we should note that the full correlation function for particle number density $\nu(\mathbf{r})$ coincides (independent of the spin projection) with $\nu(\mathbf{r}, \sigma)$.

Problem 10.32

Considering the interaction between particles as a perturbation, find the ground state energy of the Bose-gas (consisting of N spinless particles in volume V) in the first order of the perturbation theory (the interaction between particles is described by a short-range repulsive potential $U(\mathbf{r}) \ge 0$, $\mathbf{r} = \mathbf{r}_a - \mathbf{r}_b$).