Karlsruher Institut für Technologie

Institut für Theorie der Kondensierten Materie

Theorie der Kondensierten Materie II SS 2015

Prof. Dr. A. Shnirman **Blatt 3**

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Linked cluster expansion:

Consider the expectation value of the scattering matrix, which appears in the diagrammatic expansion of the Green's function $\langle 0|S|0\rangle$, where

$$S = Te^{-i\int_{-\infty}^{\infty} dt\, V_0(t)} \ .$$

Here $V_0(t)$ is the interaction Hamiltonian in the interaction representation, $V_0 = e^{iH_0t}Ve^{-iH_0t}$ and

$$V = \frac{1}{2}\int dr_1 dr_2\, \Psi^\dagger(r_1)\Psi^\dagger(r_2)V(r_1-r_2)\Psi(r_2)\Psi(r_1) \ .$$

(consider for definiteness spineless bosons)

(a) **Consider a scenario in which the interaction was switched adiabatically on around time $t_- = -T/2$ and it is switched adiabatically off around time $t_+ = T/2$. Both switching on and switching off are performed within a time interval of order δT. In order to switch on and off adiabatically, δT must be long enough, but it is still much shorter than T, i.e., $\delta T \ll T$. Relate the energy of the interacting ground state E to the energy of the non-interacting ground state E_0 via $\langle 0|S|0\rangle$.**

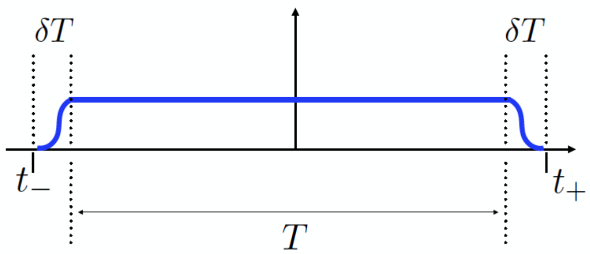

Abbildung 1: Adiabatic switching

We choose t_- to be the time just before the switching on starts and t_+ is the time right after the switching out stops (see Fig. 1). Let us split the scattering matrix as follows:

$$S = S(+\infty,-\infty) = S(+\infty,t_+)S(t_+,t_-)S(t_-,-\infty)$$

In the time period $(t_-,-\infty)$ there was no interaction, therefore $S(t_-,-\infty) = 1$. The same is valid for the time period $(+\infty,t_+)$. Thus $\langle 0|S|0\rangle = \langle 0|S(t_+,t_-)|0\rangle$. We use now

$$S(t_+,t_-) = e^{iH_0t_+}\, U_S(t_+,t_-)\, e^{-iH_0t_-} \ ,$$

where $U_S(t_+, t_-)$ is the evolution operator in the Schrödinger picture. Thus we obtain

$$\langle 0|S|0\rangle = \langle 0|U_S(t_+, t_-)|0\rangle \, e^{iE_0(t_+ - t_-)} \ .$$

We cannot calculate $\langle 0|U_S(t_+, t_-)|0\rangle$ exactly, but in the limit $\delta T \ll T$ we can argue that for most of the time period (t_+, t_-), i.e., for the whole period except δT in the beginning and δT at the end the system is in the interacting ground state. Therefore $\ln\langle 0|U_S(t_+, t_-)|0\rangle \approx -iET$. Note that it would not be correct to write $\langle 0|U_S(t_+, t_-)|0\rangle \approx e^{-iET}$. Indeed the phase ET can be large, i.e., much larger that 2π. The exponent e^{-iET} is only sensitive to $ET \mod (2\pi)$. Moreover, the phase acquired during the switching on and out could be also of order or even larger than 2π. Yet, if we would continuously follow after $\ln\langle 0|U_S(t_+, t_-)|0\rangle$ we would observe multiple phase windings and the estimate

$$\ln\langle 0|U_S(t_+, t_-)|0\rangle \approx -iET$$

becomes meaningful. Finally we obtain

$$\ln\langle 0|S|0\rangle \approx -i(E - E_0)T \ .$$

(b) **Work out, using Wick's theorem, the diagrams contributing to $\langle 0|S|0\rangle$ in the first and in the second order in V_0. Classify these into connected and non-connected diagrams. Find all topologically non-equivalent connected diagrams. Determine the multiplicity coefficients, i.e., the number of times these topologically non-equivalent diagrams contribute to $\langle 0|S|0\rangle$.**

There are no disconnected diagrams in the first order. The connected diagrams of the first order in V_0 are shown in Fig. 2. These diagrams are multiplied by $1/p$ in addition to the usual factor $i^n(\pm 1)^L$ (L is the number of loops, $n = 1$ here). The symmetry factors p are shown in Fig. 2. The symmetry factor should be understood

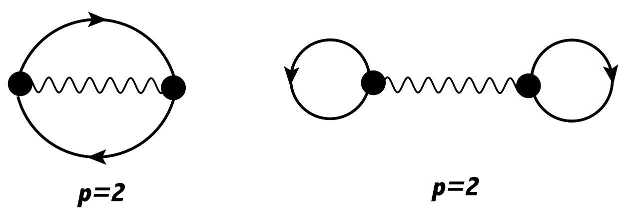

Abbildung 2: First order connected diagrams

as follows: Each "pairing" according to Wick's theorem comes with the coefficient $(2^n n!)^{-1}$. A naive argument would be to say that one has $n!$ permutations of different V_0's and 2^n permutations of the interaction "wavy" lines. Thus the total coefficient in front of each kind of diagram is 1. In the calculation of Green's function this argument works. For the diagrams of the expansion of $\langle 0|S|0\rangle$ this does not work. The reason is: some permutations lead to exactly the same "pairing". These are the symmetries, i.e., the transformations of the diagram which lead to exactly the same diagram. The number of such transformations is denoted by p. The factor $2^n n!$ should be, thus, reduced by p. As a result, each topologically unique diagram comes with a factor $1/p$. Alternatively one can explicitly count the number of different pairings leading to each topologically unique diagram. One would see that this number is $2^n n!/p$.

The connected diagrams of the second order in V_0 are shown in Fig. 3. These diagrams are multiplied by $1/p$ in addition to the usual factor $i^n(\pm 1)^L$ (L is the number of loops, $n = 2$ here). The symmetry factors p are shown in Fig. 3.

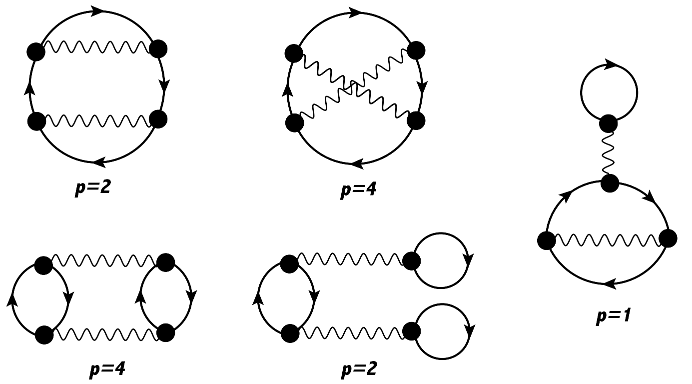

Abbildung 3: Second order connected diagrams

(c) **Argue, based on the result of (a), that the non-connected diagrams could not contribute to** $\ln\langle 0|S|0\rangle$**.**

An example of a disconnected diagram is shown in Fig. 4. This diagram clearly scales as T^2. Generally, the contributions of non-connected diagrams scale as T^n,

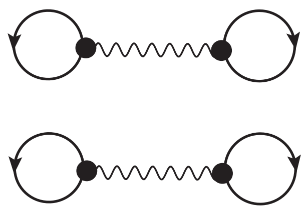

Abbildung 4: A disconnected diagram of second order

where $n > 1$ is the number of connected parts. On the other hand, as we have seen above

$$\ln\langle 0|S|0\rangle \approx -i(E - E_0)T \ .$$

Thus, only connected diagrams should contribute to the expansion of $\ln\langle 0|S|0\rangle$.

(d) **Prove the "linked cluster theorem", which states that the following expansion holds**

$$\ln\langle 0|S|0\rangle = \sum_m C_m \ ,$$

where

$$C_m = \frac{(-i)^m}{m!} \int dt_1 \dots dt_m \langle 0|TV_0(t_1)\dots V_0(t_m)|0\rangle_{conn} \ .$$

The subscript *conn* **means here "connected". Thus,** C_m **is the sum of all connected diagrams of order** m **in** V_0**. Determine the multiplicity factors, with which topologically non-equivalent diagrams contribute to** C_m**.**

Read, e.g., the book by Abrikosov, Gorkov, Dzyaloshinski (Chapter 3 Par. 15). In the book the proof is done in imaginary time. This is the same in real time.