**Theorie der Kondensierten Materie II SS 2015**

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### 1. Jordan-Wigner Transformation:

(a) Using the trivial property $(\sigma_m^z)^2 = 1$, we express the fermion operators in terms of the Pauli matrices

$$a_n = \sigma_n^- \prod_{m<n} \sigma_m^z, \quad a_n^\dagger = \sigma_n^+ \prod_{m<n} \sigma_m^z.$$

Note, that since the products do not contain any Pauli matrices at the site $n$, the products commute with $\sigma_n^\pm$. Consequently,

$$a_n^\dagger a_n = \sigma_n^+ \sigma_n^-, \quad a_n a_n^\dagger = \sigma_n^- \sigma_n^+.$$

Using explicit form of the Pauli matrices we now find

$$\sigma_n^+ \sigma_n^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_n^- \sigma_n^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus we recover the anticommutation relation

$$a_n^\dagger a_n + a_n a_n^\dagger = 1.$$

Similarly,

$$a_n^2 = \left(\sigma_n^-\right)^2 = 0, \quad \left(a_n^\dagger\right)^2 = \left(\sigma_n^+\right)^2 = 0.$$

For operators belonging to different sites, we can (without loss of generality) consider two sites $n_1 < n_2$. Then

$$a_{n_1}^\dagger a_{n_2} = \sigma_{n_1}^+ \prod_{m<n_1} \sigma_m^z \prod_{m<n_2} \sigma_m^z \, \sigma_{n_2}^- = \sigma_{n_1}^+ \sigma_{n_2}^- \prod_{m=n_1}^{n_2-1} \sigma_m^z.$$

For the opposite order of the operators we find

$$a_{n_2} a_{n_1}^\dagger = \sigma_{n_2}^- \prod_{m<n_2} \sigma_m^z \prod_{m<n_1} \sigma_m^z \, \sigma_{n_1}^+ = \sigma_{n_2}^- \prod_{m=n_1}^{n_2-1} \sigma_m^z \, \sigma_{n_1}^+ = \sigma_{n_2}^- \sigma_{n_1}^z \sigma_{n_1}^+ \prod_{m=n_1+1}^{n_2-1} \sigma_m^z$$

Now, the remaining commutation we'll perform explicitly:

$$\sigma_{n_1}^z \sigma_{n_1}^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_{n_1}^+,$$

$$\sigma_{n_1}^+ \sigma_{n_1}^z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -\sigma_{n_1}^+,$$

and hence

$$a_{n_2} a^\dagger_{n_1} = -\sigma^-_{n_2}\sigma^+_{n_1} \prod_{m=n_1}^{n_2-1} \sigma^z_m = -a^\dagger_{n_1} a_{n_2},$$

proving the anticommutation for fermionic operators on different sites. Other pairs of operators can be considered in the same fashion.

(b) Consider the product of operators on adjacent sites:

$$a^\dagger_n a_{n+1} = \sigma^+_n \prod_{m<n} \sigma^z_m \prod_{m<n+1} \sigma^z_m \, \sigma^-_{n+1} = \sigma^+_n \sigma^z_{n+1} \sigma^-_{n+1} = -\sigma^+_n \sigma^-_{n+1};$$

$$a_n a^\dagger_{n+1} = \sigma^-_n \prod_{m<n} \sigma^z_m \, \sigma^+_{n+1} \prod_{m<n+1} \sigma^z_m = \sigma^-_n \sigma^+_{n+1} \sigma^z_n = \sigma^-_n \sigma^+_{n+1};$$

$$a^\dagger_n a^\dagger_{n+1} = \sigma^+_n \prod_{m<n} \sigma^z_m \, \sigma^+_{n+1} \prod_{m<n+1} \sigma^z_m = \sigma^+_n \sigma^+_{n+1} \sigma^z_n = -\sigma^+_n \sigma^+_{n+1};$$

$$a_n a_{n+1} = \sigma^-_n \prod_{m<n} \sigma^z_m \prod_{m<n+1} \sigma^z_m \, \sigma^-_{n+1} = \sigma^-_n \sigma^z_n \sigma^-_{n+1} = \sigma^-_n \sigma^-_{n+1}.$$

Now we can use the above relations to transform the terms in the Hamiltonian:

$$\sigma^x_n \sigma^x_{n+1} = \left[\sigma^+_n + \sigma^-_n\right] \left[\sigma^+_{n+1} + \sigma^-_{n+1}\right] = -a^\dagger_n a^\dagger_{n+1} - a^\dagger_n a_{n+1} + a_n a^\dagger_{n+1} + a_n a_{n+1}$$

$$= -\left[a^\dagger_n - a_n\right] \left[a^\dagger_{n+1} + a_{n+1}\right],$$

$$\sigma^y_n \sigma^y_{n+1} = -\left[\sigma^+_n - \sigma^-_n\right] \left[\sigma^+_{n+1} - \sigma^-_{n+1}\right] = a^\dagger_n a^\dagger_{n+1} - a^\dagger_n a_{n+1} + a_n a^\dagger_{n+1} - a_n a_{n+1}$$

$$= \left[a^\dagger_n + a_n\right] \left[a^\dagger_{n+1} - a_{n+1}\right],$$

As a result, the Hamiltonian of a generic spin chain takes the form

$$\hat{H} = -\sum_{n=-\infty}^{\infty} \Big\{ J_x \left[a^\dagger_n - a_n\right] \left[a^\dagger_{n+1} + a_{n+1}\right] - J_y \left[a^\dagger_n + a_n\right] \left[a^\dagger_{n+1} - a_{n+1}\right]$$

$$-J_z \left[2a^\dagger_n a_n - 1\right] \left[2a^\dagger_{n+1} a_{n+1} - 1\right] + B \left[2a^\dagger_n a_n - 1\right] \Big\}$$

**2. Bogolyubov transformation:**

In the fermionic representation, the quantum Ising model is described by the Hamiltonian

$$\hat{H} = -\sum_{n=-\infty}^{\infty} \Big\{ J_x \left[a^\dagger_n - a_n\right] \left[a^\dagger_{n+1} + a_{n+1}\right] + B \left[2a^\dagger_n a_n - 1\right] \Big\}$$

(a) We will diagonalize the above Hamiltonian in two steps.
Firstly, we perform a Fourier transformation

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{-ikn} a_k, \quad a_n^\dagger = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{ikn} a_k^\dagger.$$

Then

$$\sum_{n=-\infty}^{\infty} a_n^\dagger a_n = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_2 e^{i(k_1-k_2)n} a_{k_1}^\dagger a_{k_2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk a_k^\dagger a_k.$$

For operators on the adjacent sites we find

$$\sum_{n=-\infty}^{\infty} a_n^\dagger a_{n+1} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_2 e^{i(k_1-k_2)n-ik_2} a_{k_1}^\dagger a_{k_2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{-ik} a_k^\dagger a_k.$$

Thus, as a result of the Fourier transform, the Hamiltonian takes the form

$$\hat{H} = -\int_{-\pi}^{\pi} \frac{dk}{2\pi} \left\{ 2(J_x \cos k + B) a_k^\dagger a_k + J_x \left( e^{ik} a_k^\dagger a_{-k}^\dagger + e^{-ik} a_{-k} a_k \right) \right\}$$

Noting the freedom in the choice of the sign of $k$, we can re-write the Hamiltonian as

$$\hat{H} = -2\int_{0}^{\pi} \frac{dk}{2\pi} \left\{ (J_x \cos k + B)\left( a_k^\dagger a_k + a_{-k}^\dagger a_{-k} \right) + iJ_x \sin k \left( a_k^\dagger a_{-k}^\dagger - a_{-k} a_k \right) \right\}$$

The integrand can now be represented in the matrix form (cf. the Nambu notation in the theory of supercondutivity)

$$\hat{H} = -2\int_{0}^{\pi} \frac{dk}{2\pi} \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} J_x \cos k + B & iJ_x \sin k \\ -iJ_x \sin k & -J_x \cos k - B \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$

Now, we need to perform the second step in the diagonalization, namely the rotation of the basis that diagonalizes the above $2 \times 2$ matrix. The excitation spectrum is given by the eigenvalues of the model

$$E_k = \sqrt{(J_x \cos k + B)^2 + J_x^2 \sin^2 k} = \sqrt{J_x^2 + B^2 + 2J_x B \cos k}$$

(b) The obtained spectrum is gapped, in the sense that the minimum value of $E_k$ is nonzero:

$$\Delta = \min\left[E_k\right] = |J_x - B|.$$

This is the minimum energy that's required to excite the lowest-lying excitation.

The gap vanishes for the special point $J_x = B$. In that case, the problem is described by a "critical" theory. Indeed, consider a change of variable $k = \pi + q$ and focus on small $q \ll 1$. Then

$$E_k \quad \rightarrow \quad E_q = \sqrt{(J_x - B)^2 + J_x B q^2}.$$

For $J_x = B$ we find the linear spectrum of massless particles

$$E_q(J_x = B) = J|q|.$$

(c) If $J_y \neq 0$, then the same Fourier transform yields

$$\hat{H} = -\int_{-\pi}^{\pi} \frac{dk}{2\pi} \left\{ 2[(J_x + J_y)\cos k + B] a_k^\dagger a_k + (J_x - J_y) \left( e^{ik} a_k^\dagger a_{-k}^\dagger + e^{-ik} a_{-k} a_k \right) \right\}$$

This is qualitatively the same problem that can be diagonalized int he same way with the only important exception: for $J_x = J_y$ the model is already diagonalized after the Fourier transform! In this case, no further rotation is needed, there is no gap in the spectrum, the excitations are the Jordan-Wigner fermions rather than Bogolyubov quasiparticles. In the spin language this special model is called the XX model and it's equivalent to free fermions as we can see from the Fourier transform. The more general XY model (where $J_x \neq J_y$) can be diagonalized by the Bogolyubov transformation and belongs to a different "universality class".