Theorie der Kondensierten Materie II SS 2017

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Lösungsvorschlag

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1. Cooper instability:

(a) The second diagram involves a bubble with the small total momentum of the two electronic states.
In the Matsubara technique, this bubble is described by the following expression

$$\Pi_C(i\omega_n, \boldsymbol{q}) = T\sum_m \int \frac{d^3p}{(2\pi)^3} \frac{1}{i\epsilon_m + i\omega_n - \xi_{\boldsymbol{p}+\boldsymbol{q}}} \frac{1}{-i\epsilon_m - \xi_{-\boldsymbol{p}}}.$$

The sum over the fermionic Matsubara frequencies can be evaluated similarly to Exercise 8.
In addition to the discussion in Exercise 8, the following basic mathematic relation can be used in evaluating such sums is

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{\psi(b)-\psi(a)}{b-a},$$

where

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \mathbf{C} + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right),$$

$\Gamma(z)$ is the Eiler Gamma function, und $\mathbf{C} = 0.577...$ is the Eiler constant. In our case, the following property of the digamma function $\psi(z)$ is important:

$$\psi\left(\frac{1}{2}+z\right) - \psi\left(\frac{1}{2}-z\right) = \pi\tan\pi z.$$

Using the above relations, we find

$$\Pi_C(i\omega_n, \boldsymbol{q}) = -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{\tanh\frac{\xi_{\boldsymbol{p}}}{2T} + \tanh\frac{\xi_{\boldsymbol{p}+\boldsymbol{q}}}{2T}}{i\omega_n - \xi_{\boldsymbol{p}} - \xi_{\boldsymbol{p}+\boldsymbol{q}}}.$$

Consider this quantity at zero frequency:

$$\Pi_C(0, \boldsymbol{q}) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{\tanh\frac{\xi_{\boldsymbol{p}}}{2T} + \tanh\frac{\xi_{\boldsymbol{p}+\boldsymbol{q}}}{2T}}{\xi_{\boldsymbol{p}} + \xi_{\boldsymbol{p}+\boldsymbol{q}}}.$$

Neglecting the momentum dependence and recalling that the attractive interaction in the BCS model exists only in a small region around the Fermi surface, we find

$$\Pi_C(0) = \frac{\nu_0}{2}\int_{-\omega_D}^{\omega_D} d\xi \frac{\tanh(\xi/2T)}{\xi} \approx \nu_0 \ln\frac{\omega_D}{T}.$$

In the last step we have used the smallness of temperature $T \ll \omega_D \ll E_F$.

Assumint that the "bare" scattering amplitude is given by a constant λ, we find that at low frequences the whole series can be written in the form of a geometric progression

$$\Gamma = \lambda + \lambda^2 \Pi_C(0) + \lambda^3 \Pi_C^2(0) + \cdots = \frac{\lambda}{1 - \lambda \Pi_C(0)}.$$

The pole in the scattering amplitude indicates an instability in the system towards formation of a new state of matter, i.e. a phase transition.

(b) Same conclusion can be reached by considering the generalized susceptibility in the Cooper channel. The diagrams for the susceptibility can be found by "closing" the outer lines in the diagrams for the scattering amplitude. The series has the form

$$\chi_C(i\omega_n) = \Pi_C(i\omega_n) + \lambda \Pi_C^2(i\omega_n) + \cdots = \frac{\Pi_C(i\omega_n)}{1 - \lambda \Pi_C(i\omega_n)}.$$

This is the same pole as above. Focusing on the zero-frequency terms, we can find the transition temperature as

$$1 - \lambda \Pi_C(0) = 0 \quad \Rightarrow \quad T_c \propto \omega_D e^{-1/(\nu_0 \lambda)}.$$

The proportionality coefficient should be found by a more precise calculation of $\Pi_C(0)$. Our above result was obtained with the "logarithmic accuracy", which means that we have neglected possible numerical factors under the logarithm. These factors would then translate into the coefficient in the expression for T_c.

(c) In the case, where the "bare" scattering amplitude is a function of frequencies (instead of a constant), one can find an integral equation for the scattering amplitude, that generalizes the above geometric series. Neglecting momentum dependencies (and setting the total frequency and momentum of the Cooper pairs to zero), we find

$$\Gamma^C(i\epsilon, i\epsilon') = \Gamma^{(0)}(i\epsilon, i\epsilon') + T \sum_{\epsilon''} \int \frac{d^3p}{(2\pi)^3} \Gamma^{(0)}(i\epsilon, i\epsilon'') G(i\epsilon'', \boldsymbol{p}) G(-i\epsilon'', -\boldsymbol{p}) \Gamma^C(i\epsilon'', i\epsilon').$$

Since the only the two Green's functions are momentum-dependent, we can evaluate the momentum integral

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{(\epsilon'')^2 + \xi_{\boldsymbol{p}}^2} = \frac{\pi \nu_0}{|\epsilon''|}.$$

As a result, the integral equation reads

$$\Gamma^C(i\epsilon, i\epsilon') = \Gamma^{(0)}(i\epsilon, i\epsilon') + \pi \nu_0 T \sum_{\epsilon''} \frac{1}{|\epsilon''|} \Gamma^{(0)}(i\epsilon, i\epsilon'') \Gamma^C(i\epsilon'', i\epsilon').$$

(d) For a factorized bare scattering amplitude, the above integral equation can be solved as follows. Substituting the explicit form of the bare scatterign amplitude, we find

$$\Gamma^C(i\epsilon, i\epsilon') = \lambda v(i\epsilon) \left[v(i\epsilon') + \pi \nu_0 T \sum_{\epsilon''} \frac{v(i\epsilon'')}{|\epsilon''|} \Gamma^C(i\epsilon'', i\epsilon') \right].$$

Clearly, $\Gamma^C(i\epsilon, i\epsilon') \propto v(i\epsilon)$. On symmetry grounds, we introduce an *Ansatz*

$$\Gamma^C(i\epsilon, i\epsilon') = \alpha v(i\epsilon) v(i\epsilon'),$$

and find the remaining constant from the integral equation:

$$\alpha = \frac{\lambda}{1 - \lambda\pi\nu_0 T \sum_{\epsilon''} \frac{v^2(i\epsilon'')}{|\epsilon''|}}.$$

This above result exhibits a pole at

$$\pi T \sum_{\epsilon''} \frac{v^2(i\epsilon'')}{|\epsilon''|} = \frac{1}{\nu_0 \lambda}.$$

This condition defines the transition temperature, similarly to the earlier considerations.

The actual values of T_c canbe obtained from the above condition by assuming a particular form of $v(i\epsilon)$. If

$$v(i\epsilon) = \frac{\omega_D}{\sqrt{\omega_D^2 + \epsilon^2}},$$

then the equation for the critical temperature becomes

$$2\sum_{n=0}^{\infty} \frac{\omega_D^2}{\left[\omega_D^2 + \pi^2 T_c^2 (2n+1)^2\right](2n+1)} = \frac{1}{\nu_0 \lambda}.$$

With the logarithmic accuracy, we can solve this equation as follows. The first term in the deniminator decays for $n \gg n^* = \omega_D/(\pi T_c)$. If we assume $n^* \gg 1$, then the sum can be approximated by

$$2\sum_{n=0}^{\infty} \frac{\omega_D^2}{\left[\omega_D^2 + \pi^2 T_c^2 (2n+1)^2\right](2n+1)} \approx \sum_{n=0}^{n^*} \frac{1}{n + 1/2} = \ln n^*.$$

The resulting T_c is

$$T_c \approx \frac{\omega_D}{\pi} e^{-1/(\nu_0 \lambda)},$$

which agrees with the previous result.