

Theorie der Kondensierten Materie II SS 2017

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Blatt 2
Lösungsvorschlag

1. Scattering in 2D: the logarithm and the renormalization group

(a) Derivative of the scattering amplitude $\partial F(\epsilon)/\partial \epsilon$.

From the last exercise sheet we know the formal expansion of the scattering amplitude $F = V + VG_0V + VG_0VG_0V + \dots$. The only quantity that depends on energy is the Green's function G_0 . If we differentiate F with respect to ϵ we need to use the product rule. It is important here to keep the correct order of the operators. We find:

$$\begin{aligned} \frac{\partial F}{\partial \epsilon} = & V \frac{\partial G_0}{\partial \epsilon} V + V \left[\frac{\partial G_0}{\partial \epsilon} V G_0 + G_0 V \frac{\partial G_0}{\partial \epsilon} \right] V \\ & + V \left[\frac{\partial G_0}{\partial \epsilon} V G_0 V G_0 + G_0 V \frac{\partial G_0}{\partial \epsilon} V G_0 + G_0 V G_0 V \frac{\partial G_0}{\partial \epsilon} \right] V + \dots \end{aligned}$$

Graphically, this can be depicted as follows, where lines with crosses correspond to the derivatives of the Green's function.

$$\frac{dF}{dE} = \text{diagrammatic expansion with crosses on } G_0 \text{ lines}$$

(b) Summing of the series and derivation of the renormalization group (RG) equation.

We can now sort all terms according to the occurrence of the position of the derivative. We first collect all terms where the derivative acts on the most left G_0 and add the terms, where the derivative acts on the second G_0 and so on:

$$\begin{aligned} \frac{\partial F}{\partial \epsilon} = & V \frac{\partial G_0}{\partial \epsilon} [V + VG_0V + \dots] + VG_0V \frac{\partial G_0}{\partial \epsilon} [V + VG_0V + \dots] \\ & + VG_0VG_0V \frac{\partial G_0}{\partial \epsilon} [V + VG_0V + \dots] + \dots \end{aligned}$$

The square brackets can now be replaced by F :

$$\begin{aligned} \frac{\partial F}{\partial \epsilon} = & V \frac{\partial G_0}{\partial \epsilon} F + VG_0V \frac{\partial G_0}{\partial \epsilon} F + VG_0VG_0V \frac{\partial G_0}{\partial \epsilon} F + \dots \\ = & F \frac{\partial G_0}{\partial \epsilon} F \end{aligned}$$

This corresponds to the diagram

$$\frac{dF}{dE} = \text{Diagram: two blue wavy lines connected by a blue line with a crossbar, representing a self-energy correction to a propagator.}$$

After computing the derivative of G_0 , we can write down the equation for the function $F(\epsilon, \mathbf{k}_1, \mathbf{k}_2)$ as an integral equation

$$\frac{\partial F(\epsilon, \mathbf{k}_1, \mathbf{k}_2)}{\partial \epsilon} = - \int \frac{d^2 q}{(2\pi)^2} \frac{F(\epsilon, \mathbf{k}_1, \mathbf{q}) F(\epsilon, \mathbf{q}, \mathbf{k}_2)}{\left(\epsilon - \frac{q^2}{2m} + i0\right)^2}.$$

So far, our calculations were exact.

Consider now the limit of low energies, $|\epsilon| \ll 1/2ma^2$, where a is the effective interaction radius of the short-range potential.

Since the scattering amplitude as a function of its momenta varies on the scale $k_{1,2} \sim 1/a$, we may disregard this dependence in the integral, since the integral converges already at small momenta $q \ll 1/a$. We assume here as well that the potential is rotationally symmetric such that there is no dependence on the the direction of the momenta. The integration over \mathbf{q} is now elementary and we find

$$\frac{\partial F}{\partial \epsilon} = \frac{m}{2\pi\epsilon} F^2(\epsilon).$$

(c) Solution of the RG equation.

The above equation can be easily solved:

$$F(\epsilon) = -\frac{2\pi}{m \ln(\epsilon/\epsilon_0 + i0)}. \quad (1)$$

Here the constant $\epsilon_0 < 0$ is real and negative. This follows from the fact, that the scattering amplitude is real for negative energies. To reach this conclusion, one can write down the perturbation series in the coordinate representation where the Green's function has the form

$$G_\epsilon(r) = -\frac{m}{2\pi r} e^{i\kappa r}, \quad \kappa^2 = 2m\epsilon,$$

which is real for negative energies.

If the constant ϵ_0 belongs to the low energy region, $\epsilon_0 < 1/(ma^2)$, then the pole $\epsilon = \epsilon_0$ has the physical meaning of the bound state. Otherwise, the pole is meaningless, since it lies beyond the energy region where our approximations are justified.

2. Polarizability of a ground state

(a) Definition of the Green's function in the momentum representation

From the first exercise sheet we know that we can write down the Green's function as $G = G_0 + G_0 F G_0$. The Green's function in the energy-momentum space can thus be written as

$$G(\epsilon, \mathbf{p}_1, \mathbf{p}_2) = \frac{(2\pi)^2 \delta(\mathbf{p}_1 - \mathbf{p}_2)}{\epsilon - \epsilon_{\mathbf{p}_1} + i0} + \frac{F(\epsilon)}{(\epsilon - \epsilon_{\mathbf{p}_1} + i0)(\epsilon - \epsilon_{\mathbf{p}_2} + i0)}$$

The ground state corresponds to the filled bound state with the (negative) energy ϵ_0 . All other states ("scattering states" at positive energies $\epsilon = p^2/(2m)$) are empty. We shift the empty states to the lower half-plane, hence the imaginary part $+i0$ in the Green's function. The pole at ϵ_0 belongs to the scattering amplitude. This pole we shift to the upper half-plane. The branch point $\epsilon = 0$ we keep in the lower half-plane.

If we calculate for example the density [or the dipole moment, see part (b)], we close the contour in the upper half-plane, which takes into account only the characteristics of the bound state.

(b) **Relation between dipole moment and exact Green's function.**

The dipole moment is given as usual by the first moment of the (effective) charge distribution

$$\begin{aligned} \mathbf{P} &= e \int d^2 r \mathbf{r} n(\mathbf{r}) \\ &= -ie \int d^2 r \mathbf{r} \lim_{t \rightarrow 0^-} G(t, \mathbf{r}, \mathbf{r}) \\ &= e \int \frac{d^2 p_1 d^2 p_2}{(2\pi)^4} \int d^2 r \mathbf{r} e^{i(\mathbf{p}_1 - \mathbf{p}_2)\mathbf{r}} \lim_{t \rightarrow 0^-} \int \frac{d\omega}{2\pi i} e^{-i\omega t} G(\omega, \mathbf{p}_1, \mathbf{p}_2) \\ &= -ie \int \frac{d^2 p_1 d^2 p_2}{(2\pi)^2} [\nabla_{\mathbf{p}_1} \delta(\mathbf{p}_1 - \mathbf{p}_2)] \sum_{\text{Im}(\omega) > 0} \text{Res}[G(\omega, \mathbf{p}_1, \mathbf{p}_2), \omega] \\ &= ie \int \frac{d^2 p_1}{(2\pi)^2} [\text{Res}(\nabla_{\mathbf{p}_1} G(\omega, \mathbf{p}_1, \mathbf{p}_2), \omega = \epsilon_0 + i0)]_{\mathbf{p}_2 = \mathbf{p}_1} \end{aligned}$$

Here we need to set $\mathbf{p}_2 = \mathbf{p}_1$ after taking the derivative w.r.t. \mathbf{p}_1 . We easily observe that without an electric field the polarization is zero ($\nabla_{\mathbf{p}_1} \epsilon_{\mathbf{p}_1} \propto \mathbf{p}_1 \Rightarrow$ odd function).

(c) **Diagrammatic expansion of the Green's function in the applied electric field.**

We now add an electric field to our system that distorts the (effective) charge distribution and leads to a finite polarization. With the help of the Green's function G that already incorporates the effects of the short-range potential, we can expand the Green's function G_W in powers of the electric potential W as $G_W = G + GWG + GWGWG + \dots$. For the Green's function G we have the equation [see part (a)] $G = G_0 + G_0 F G_0$. Diagrammatically this looks like

For the polarizability, we need the residue of the Green's function.

The residue of the Green's function G_W can be obtained by plugging ϵ_0 into the free Green's function G_0 and calculating the residue of F :

$$\sum_{\omega} \text{Res}G(\omega, \mathbf{p}, \mathbf{p}') = \text{Res}F(\epsilon) \Big|_{\epsilon_0} \frac{ie}{m} E [p'_x G_0(\epsilon_0, \mathbf{p}) G_0^3(\epsilon_0, \mathbf{p}') - p_x G_0^3(\epsilon_0, \mathbf{p}) G_0(\epsilon_0, \mathbf{p}')].$$

The two terms contribute equally due to symmetry.

The residue of the scattering amplitude can be found as follows

$$F(\epsilon) = -\frac{2\pi}{m \ln(\epsilon/\epsilon_0)} \stackrel{\epsilon \rightarrow \epsilon_0}{\sim} \frac{-2\pi}{m/\epsilon_0(\epsilon - \epsilon_0)} \quad \Rightarrow \quad \text{Res}F(\epsilon) \Big|_{\epsilon_0} = -\frac{2\pi\epsilon_0}{m}$$

The polarizability χ measures the amount of polarization for a weak electric field: $\mathbf{P} = \chi \mathbf{E}$. In this simple example the tensor χ is diagonal. The applied field induces a polarization in the same direction (here: x -direction). We find

$$\chi = ie \frac{\partial}{\partial E} \int \frac{d^2p}{(2\pi)^2} \left[\frac{\partial}{\partial p_x} \text{Res}(\delta G(\omega, \mathbf{p}, \mathbf{p}')) \right]_{\mathbf{p}'=\mathbf{p}, E \rightarrow 0}.$$

Pictorially, the polarizability can be represented by

$$\chi = \text{---} \bullet \text{---} \delta G$$

where the Green's function contains all of the above four terms and the dashed line corresponds to either $\nabla\delta(\mathbf{r})$ (in momentum representation) or \mathbf{r} (in coordinate representation). Then the first term in δG vanishes after the integration over energies, the fourth due to the odd character of \mathbf{r} , while the second and third are identical due to symmetry. All remaining integrals are elementary (we close the contour in the upper-half plane, such that only the pole at ϵ_0 contributes), the final result reads

$$\chi = \frac{e^2}{6m\epsilon_0^2}.$$

Interpretation: A large charge e and a small binding energy ϵ_0 lead to a large polarizability. This is not surprising since the electric potential energy is proportional to the charge and one expects the electron to be more susceptible to perturbations the weaker it is bound by the short-range potential.