Karlsruher Institut für Technologie Institute for Condensed Matter Theory
Institute for Quantum Materials and Technologies

Condensed Matter Theory II: Many-Body Theory (TKM II) SoSe 2022

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Sheet 0
Live exercise on 21 April 2022

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1. Complex contour integrals

(? Bonus points)

Calculate the following integrals using the techniques discussed during class.

(a) $I_1 = \oint_{|z|=2} \mathrm{d}z \, \frac{z}{\exp(\mathrm{i}z)-1}$
Hint: Use the Cauchy theorem.

Solution: The integrand is holomorphic everywhere but at $z = 2\pi k$, $k \in \mathbb{Z}\backslash\{0\}$. (The point $z = 0$ can be considered by expanding the denominator around $z = 0$.) None of these points is enclosed by the contour, thus

$$I_1 = 0 \tag{1}$$

(b) $I_2 = \oint_{|z|=2} \mathrm{d}z \exp\left(\frac{z}{z-1}\right)$
Hint: Use the residue theorem, calculating the residue via Laurent expansion.

Solution: The integrand is holomorphic everywhere, but at $z = 1$. At this point, the function has an essential singularity, which does not correspond to a finite order pole. Thus, it is most convenient, to calculate the residuum directly via its definition as the -1 coefficient in the Laurent expansion.
We find the Laurent expansion of the exponential function by substituting $x = z - 1$ and performing a Taylor expansion around $x = 0$. This expansion is valid everywhere but at $z = 1$:

$$\exp\left(\frac{x+1}{x}\right) = \exp(1)\exp\left(\frac{1}{x}\right) \tag{2}$$

$$= \mathrm{e}\sum_{n=0}^{\infty} \frac{x^{-n}}{n!} \tag{3}$$

$$= \mathrm{e}\sum_{n=0}^{\infty} \frac{(z-1)^{-n}}{n!} \tag{4}$$

$$= \mathrm{e}\sum_{n=-\infty}^{0} \frac{(z-1)^{n}}{(-n)!} \tag{5}$$

The -1 coefficient of this expansion (corresponding to $1/(z-1)$) is e. Thus, according to the residue theorem

$$I_2 = \mathrm{i}2\pi\mathrm{e}. \tag{6}$$

Using mathematica, we can verify this by parametrizing the contour as $z = 2\exp(\mathrm{i}\phi)$, $\phi \in [0, 2\pi)$.

(c) $I_3 = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x\sin(\alpha x)}{x^2+\beta^2}$, with real numbers $\alpha, \beta > 0$.
Hint: Find a way to rewrite this integral in terms of a complex contour integral. Solve the resulting integral using the residue theorem.
Solution:

$$
\begin{aligned}
I_3 &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x\sin(\alpha x)}{x^2+\beta^2} && (7)\\
&= \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x(\exp(\mathrm{i}\alpha x) - \exp(-\mathrm{i}\alpha x))}{x^2+\beta^2} && (8)\\
&= \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x\exp(\mathrm{i}\alpha x)}{x^2+\beta^2} - \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x\exp(-\mathrm{i}\alpha x)}{x^2+\beta^2} && (9)\\
&= -\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x\exp(\mathrm{i}\alpha x)}{x^2+\beta^2} && (10)\\
&= -\mathrm{i} \oint_{\gamma} \mathrm{d}x \, \frac{x\exp(\mathrm{i}\alpha x)}{x^2+\beta^2} + \underbrace{\mathrm{i} \int_{\text{half circle}} \mathrm{d}x \, \frac{x\exp(\mathrm{i}\alpha x)}{x^2+\beta^2}}_{:=I_3'} && (11)
\end{aligned}
$$

Here, γ is a path along the real axis, which is closed by a half circle of radius $R \to \infty$ in the upper complex half plane. By showing, that the integral over the half circle I_3' vanishes, we can compute I_3 via the integral over the closed contour γ, which can be solved using the residue theorem.
First, we show that the half circle integral vanishes. (We can also see this directly invoking **Jordan's lemma**.) To this end, we parametrize the contour as $z = R\exp(\mathrm{i}\phi)$, $\phi \in (0, \pi)$:

$$
I_3' = \mathrm{i} \lim_{R\to\infty} R^2 \int_0^{\pi} \mathrm{d}\phi \exp(2\mathrm{i}\phi) \frac{\exp(\mathrm{i}\alpha R\exp(\mathrm{i}\phi))}{R^2\exp(2\mathrm{i}\phi) + \beta^2} \tag{12}
$$

We consider the modulus:

$$
\begin{aligned}
|I_3'| &\le \lim_{R\to\infty} R^2 \int_0^{\pi} \mathrm{d}\phi \, \frac{|\exp(\mathrm{i}\alpha R[\cos(\phi) + \mathrm{i}\sin(\phi)])|}{\sqrt{R^4+\beta^4+2R^2\beta^2\cos(2\phi)}} && (13)\\
&= \lim_{R\to\infty} R^2 \int_0^{\pi} \mathrm{d}\phi \, \frac{\exp(-\alpha R\sin(\phi))}{\sqrt{R^4+\beta^4+2R^2\beta^2\cos(2\phi)}} && (14)\\
&= \lim_{R\to\infty} R^2 \left[\int_{\delta}^{\pi-\delta} \mathrm{d}\phi \, \frac{\exp(-\alpha R\sin(\phi))}{\sqrt{R^4+\beta^4+2R^2\beta^2\cos(2\phi)}} + 2\int_0^{\delta} \mathrm{d}\phi \, \frac{\exp(-\alpha R\sin(\phi))}{\sqrt{R^4+\beta^4+2R^2\beta^2\cos(2\phi)}} \right] && (15)
\end{aligned}
$$

In the last step we subdivided the integral into two three regions ($[0,\delta]$ gives the same result as $[\pi-\delta,\pi]$), which we consider separately. The first term we estimate by the interval length times the maximum of the integrand:

$$
|I_3'| \le \lim_{R\to\infty} R^2 \left[\frac{\pi\exp(-\alpha R\sin(\delta))}{\sqrt{R^4+\beta^4+2R^2\beta^2\cos(2\delta)}} + 2\int_0^{\delta} \mathrm{d}\phi \, \frac{\exp(-\alpha R\sin(\phi))}{\sqrt{R^4+\beta^4+2R^2\beta^2\cos(2\phi)}} \right] \tag{16}
$$

In the second term we use $\exp(-\alpha R\sin(\phi)) \le 1$:

$$
|I_3'| \le \lim_{R\to\infty} R^2 \left[\frac{\pi\exp(-\alpha R\sin(\delta))}{\sqrt{R^4+\beta^4+2R^2\beta^2\cos(2\delta)}} + 2\delta \frac{1}{\sqrt{R^4+\beta^4}} \right] \tag{17}
$$

Setting $\delta := 1/\sqrt{R}$, the integral vanishes as $I_3' \sim 1/R$ for $R \to \infty$. Note, that all of this would not work for a higher power of x in the numerator of the integrand in I_3.
As $I_3' = 0$, we find

$$I_3 = -\mathrm{i} \oint_\gamma \mathrm{d}x \, \frac{x \exp(\mathrm{i}\alpha x)}{x^2 + \beta^2} \tag{18}$$

The integrand is holomorphic everywhere, but at $x = \pm \mathrm{i}\beta$:

$$\frac{1}{(x^2 + \beta^2)} = \frac{1}{x + \mathrm{i}\beta} \frac{1}{x - \mathrm{i}\beta} \tag{19}$$

The first order pole at $x = \mathrm{i}\beta$ is enclosed by γ, therefore

$$I_3 = 2\pi \mathrm{Res}_{z_0 = \mathrm{i}\beta} \left[\frac{x \exp(\mathrm{i}\alpha x)}{x^2 + \beta^2} \right] \tag{20}$$

$$= 2\pi \exp(-\alpha\beta)/2. \tag{21}$$

(d) $I_4 = \int_{-1}^{1} \mathrm{d}x \, (1+x)^\alpha (1-x)^{1-\alpha}$, $0 < \alpha < 1$
Hint: The integrand contains a branch cut in the complex plane. Find this branch cut, and show that the integral can be expressed as

$$I_4 = \mathrm{const} \cdot \oint_{\mathcal{C}} \mathrm{d}z \, (1+z)^\alpha (z-1)^{1-\alpha} \tag{22}$$

where $\mathcal{C}$ is an appropriately chosen contour enclosing the line $[-1, 1]$. Determine the constant and solve the resulting integral using a substitution $z \to 1/w$.
Solution: The initial integrand is analytic everywhere except at the branch cuts $]-\infty, -1]$ and $[1, \infty[$ on the real axis. We can shift the branch cut to $[-1, 1]$ by modifying the second term.
By shifting x by infinitesimal ε to upper/lower half-plane (UHP/LHP), we write

$$(1-x)^{1-\alpha} = [(x + i\varepsilon - 1)(-1 - i\varepsilon)]^{1-\alpha} = \exp[-(1-\alpha)\pi i](x + i\varepsilon - 1)^{1-\alpha}, \tag{23}$$

where we used the exponential identity

$$(ab)^\alpha = a^\alpha b^\alpha, \tag{24}$$

which is only valid when the sum of the complex phases of a and b do not cross a branch cut:

$$-\pi < \mathrm{ph}\, a + \mathrm{ph}\, b < \pi. \tag{25}$$

This implies that when we shift $x - 1$ to UHP, we must shift -1 to LHP, or vice versa.
Since $(1-x)^{1-\alpha}$ is real for $-1 < x < 1$, we may take the absolute value:

$$(1-x)^{1-\alpha} = |(x + i\varepsilon - 1)^{1-\alpha}| = \frac{\mathrm{Im}\left[(x + i\varepsilon - 1)^{1-\alpha}\right]}{\sin[(1-\alpha)\pi]}, \tag{26}$$

on the second step we used the imaginary part of the identity

$$|w| e^{i \mathrm{ph}\, w} = \mathrm{Re}\, w + i \mathrm{Im}\, w. \tag{27}$$

Similarly, the shifting x to LHP gives

$$(1-x)^{1-\alpha} = -\frac{\operatorname{Im}\left[(x-i\varepsilon-1)^{1-\alpha}\right]}{\sin(\alpha\pi)}. \tag{28}$$

Now the integral can be expressed as a contour integral

$$I_4 = -\frac{1}{2\sin(\alpha\pi)}\operatorname{Im}\oint_\gamma dz(1+z)^\alpha(z-1)^{1-\alpha}, \tag{29}$$

where the closed contour γ goes first from $-1-i\varepsilon$ to $1-i\varepsilon$ and then from $1+i\varepsilon$ to $-1+i\varepsilon$. The connecting infinitesimal parts can be neglected, since the integrand is finite.

The above form still does not allow us to use the residue theorem, since there is a branch cut on the real axis for $z\in[-1,1]$. The usual trick is to transform

$$z = \frac{1}{w}. \tag{30}$$

Using this we obtain

$$-\oint_\gamma dw\frac{(1+w^{-1})^\alpha(w^{-1}-1)^{1-\alpha}}{w^2} = -\oint_\gamma dw\frac{(w+1)^\alpha(1-w)^{1-\alpha}}{w^3} \tag{31}$$

as the integrand. The numerator is analytic everywhere but the branch cuts. The branch cut is transformed to two cuts at $]-\infty,-1]$ and $[1,\infty[$ and the transformed contour avoids it (see Fig. 1 for a picture of the contour after the transformation). The only singularity inside the contour is at $w=0$ and comes from the denominator.

The Taylor expansion of the numerator at $w=0$ is

$$(w+1)^\alpha(1-w)^{1-\alpha} = 1-(1-2\alpha)w-2\alpha(1-\alpha)w^2+\mathcal{O}\left(w^3\right). \tag{32}$$

Multiplying this with the denominator w^{-3}, the -1^{th} Laurent coefficient of the integrand can be identified as

$$a_{-1} = -2\alpha(1-\alpha). \tag{33}$$

After using the residue theorem, taking the imaginary part and adding the prefactor, we find that the integral is

$$I_4 = \frac{2\pi\alpha(1-\alpha)}{\sin(\alpha\pi)}. \tag{34}$$

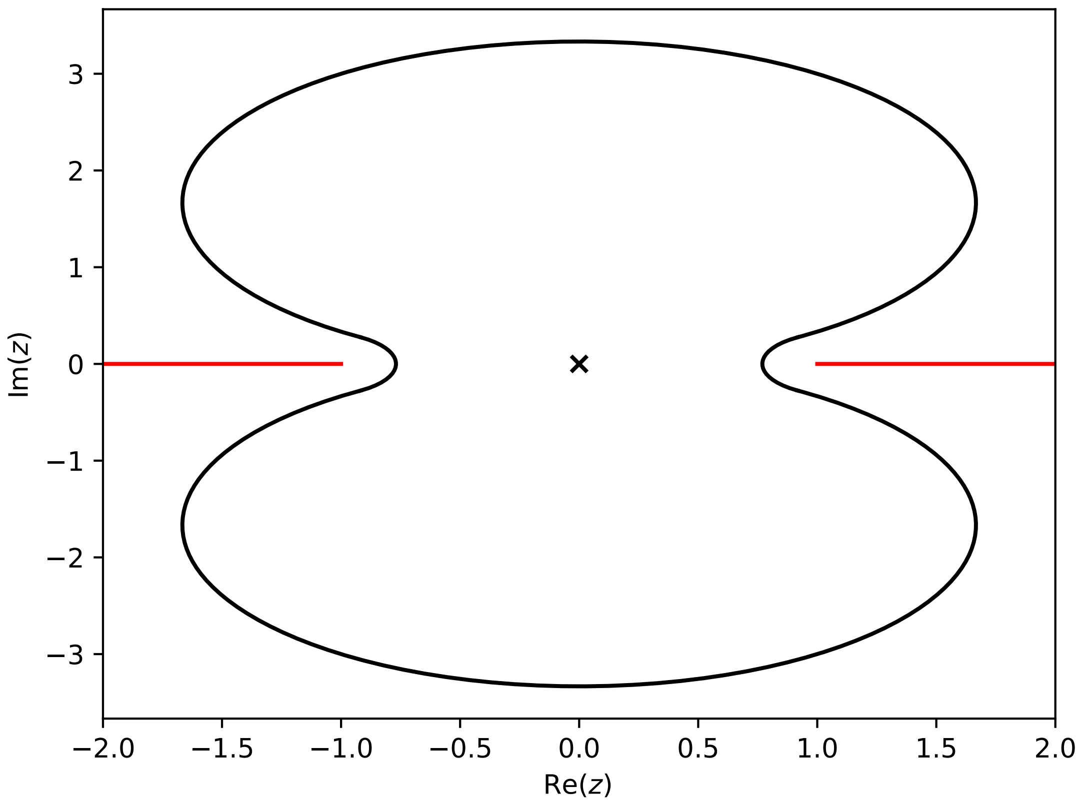

Figure 1: The box contour (black line) for I_4 after transformation $w = 1/z$. Red lines are branch cuts, x marks the singularity within the integration path. For this example, $\varepsilon = 0.3$ was chosen.