Condensed Matter Theory II: Many-Body Theory (TKM II) SoSe 2023

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Homework assignment 2
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1. Green's function of phonons (20 points)

In the lectures, when addressing the many-body Green's functions, we mainly focused on fermions. Here, we will discuss bosons, using phonons as an example. Consider flexural phonons with the Hamilton operator

$$\widehat{H} = \sum_{\mathbf{q}} \omega_q \left(\hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + \frac{1}{2} \right),$$

where $\omega_q = \kappa |\mathbf{q}|^2$, κ is the lattice stiffness, and $\mathbf{q}$ is a 2D momentum. Introduce the field operator,

$$\widehat{\Phi}(\mathbf{r}) = \mathrm{i} \sum_{\mathbf{q}} \sqrt{\frac{\omega_q}{2V}} \left(\hat{b}_{\mathbf{q}} e^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}} - \hat{b}^\dagger_{\mathbf{q}} e^{-\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}} \right).$$

Determine the Green's function of phonons in the $\mathbf{q}, \omega$-representation. Then Fourier transform the result to $\mathbf{r}, t$-representation, assuming a momentum-cutoff at $q = \Lambda$.

2. Polarizability of a particle in a 1D potential ($8 + 10 + 12$ points)

Consider a charged particle in a one-dimensional system with a potential well characterized by the amplitude V_0 and the spatial range a. Assume that $V_0 \ll \hbar^2/(2ma^2)$. Suppose the particle is in the ground state. The polarizability χ in a weak external electric field $\mathbf{E}$ relates the polarization (dipole moment) with the field: $\mathbf{P} = \chi \mathbf{E}$.

(a) Write down the expression for the Green's function in the momentum representation in the absence of electric field in terms of the scattering amplitude F. Solve the equation for $F(\varepsilon, p_1 \approx 0, p_2 \approx 0)$ when $V_0 \ll \hbar^2/(2ma^2)$ by assuming that F does not have any poles in momentum space. Estimate the bound state energy ϵ_0.

(b) Express the dipole moment $P = \int \mathrm{d}x\, x n(x)$ of the system in terms of the exact Green's function in the momentum representation. You should find

$$P = ie \int \frac{dp_1}{2\pi} \left[\frac{\partial}{\partial p_1} \mathrm{Res} G^R(\varepsilon, p_1, p_2) \right] \Bigg|_{\varepsilon = \epsilon_0 + i0; p_1 = p_2}.$$

(c) Consider now the Green's function in the potential

$$W = -eEx$$

induced by the applied electric field. What is a graphical representation for the linear-in-E correction to the Green's function? Using the diagrams, evaluate the polarizability of the system.

Hint: Why can you use F with $p_1 \approx 0$, $p_2 \approx 0$?