Karlsruher Institut für Technologie Institute for Condensed Matter Theory
Institute for Quantum Materials and Technologies

Condensed Matter Theory II: Many-Body Theory (TKM II) SoSe 2023

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Homework assignment 2
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1. Green's function of phonons (20 points)

In the lectures, when addressing the many-body Green's functions, we mainly focused on fermions. Here, we will discuss bosons, using phonons as an example. Consider flexural phonons with the Hamilton operator

$$\widehat{H} = \sum_{\mathbf{q}} \omega_q \left(\hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + \frac{1}{2} \right),$$

where $\omega_q = \kappa |\mathbf{q}|^2$, κ is the lattice stiffness, and $\mathbf{q}$ is a 2D momentum. Introduce the field operator,

$$\widehat{\Phi}(\mathbf{r}) = \mathrm{i} \sum_{\mathbf{q}} \sqrt{\frac{\omega_q}{2V}} \left(\hat{b}_{\mathbf{q}} e^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}} - \hat{b}^\dagger_{\mathbf{q}} e^{-\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}} \right).$$

Determine the Green's function of phonons in the $\mathbf{q}, \omega$-representation. Then Fourier transform the result to $\mathbf{r}, t$-representation, assuming a momentum-cutoff at $q = \Lambda$.

Solution:

Heisenberg representation:

$$\widehat{\Phi}(\mathbf{r}, t) = \mathrm{i} \sum_{\mathbf{q}} \sqrt{\frac{\omega_q}{2V}} \left(\hat{b}_{\mathbf{q}} e^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{r} - \mathrm{i}\omega_{\mathbf{q}} t} - \hat{b}^\dagger_{\mathbf{q}} e^{-\mathrm{i}\,\mathbf{q}\cdot\mathbf{r} + \mathrm{i}\omega_{\mathbf{q}} t} \right).$$

Definition of Green's function [note that the phonon field is real: $\widehat{\Phi}^\dagger(\mathbf{r}, t) = \widehat{\Phi}(\mathbf{r}, t)$]:

$$D(\mathbf{r}, t; \mathbf{r}', t) = -i \langle 0 | \mathcal{T} \widehat{\Phi}(\mathbf{r}, t) \widehat{\Phi}(\mathbf{r}', t') | 0 \rangle, \tag{1}$$

where $|0\rangle$ denotes the ground state (no phonons). Translational invariance in space and time:

$$D(\mathbf{r}, t; \mathbf{r}', t') \to D(\mathbf{r} - \mathbf{r}', t - t') \to D(\mathbf{r}, t). \tag{2}$$

$$\begin{aligned}
&\langle 0 | b^\dagger_{\mathbf{q}} b_{\mathbf{q}'} | 0 \rangle = \langle 0 | b^\dagger_{\mathbf{q}} b^\dagger_{\mathbf{q}'} | 0 \rangle = \langle 0 | b_{\mathbf{q}} b_{\mathbf{q}'} | 0 \rangle = 0, \qquad \langle 0 | b_{\mathbf{q}} b^\dagger_{\mathbf{q}'} | 0 \rangle = \delta_{\mathbf{q}\mathbf{q}'} \qquad \Longrightarrow \\
&\langle 0 | (\alpha b_{\mathbf{q}} + \beta b^\dagger_{\mathbf{q}})(\alpha' b_{\mathbf{q}'} + \beta' b^\dagger_{\mathbf{q}'}) | 0 \rangle = \langle 0 | \alpha \beta' b_{\mathbf{q}} b^\dagger_{\mathbf{q}'} | 0 \rangle = \alpha \beta' \delta_{\mathbf{q}\mathbf{q}'}, \\
&\langle 0 | (\alpha' b_{\mathbf{q}'} + \beta' b^\dagger_{\mathbf{q}'})(\alpha b_{\mathbf{q}} + \beta b^\dagger_{\mathbf{q}}) | 0 \rangle = \langle 0 | \alpha' \beta b_{\mathbf{q}'} b^\dagger_{\mathbf{q}} | 0 \rangle = \alpha' \beta \delta_{\mathbf{q}\mathbf{q}'}.
\end{aligned} \tag{3}$$

$t > 0$:

$$D(\mathbf{r}, t) \;=\; -i \sum_{\mathbf{q}} \left(\sqrt{\frac{\omega_{\mathbf{q}}}{2V}} \right)^2 \langle 0 | b_{\mathbf{q}} e^{i\mathbf{q}\mathbf{r} - i\omega_{\mathbf{q}} t} b^\dagger_{\mathbf{q}} | 0 \rangle = -i \sum_{\mathbf{q}} \frac{\omega_{\mathbf{q}}}{2V} e^{i\mathbf{q}\mathbf{r} - i\omega_{\mathbf{q}} t}. \tag{4}$$

$t < 0$:

$$D(\mathbf{r},t) = -i\sum_{\mathbf{q}}\left(\sqrt{\frac{\omega_{\mathbf{q}}}{2V}}\right)^2 \langle 0|b_{\mathbf{q}}e^{-i\mathbf{q}\mathbf{r}+i\omega_{\mathbf{q}}t}b^\dagger_{\mathbf{q}}|0\rangle = -i\sum_{\mathbf{q}}\frac{\omega_{\mathbf{q}}}{2V}e^{-i\mathbf{q}\mathbf{r}+i\omega_{\mathbf{q}}t}. \tag{5}$$

Fourier transformation from $\mathbf{r}, t$ to $\mathbf{q}, \omega$:

$$\begin{aligned}
D(\mathbf{q},\omega) &= -i\frac{\omega_{\mathbf{q}}}{2}\left[\int_{-\infty}^{0} dte^{i\omega t+i\omega_{\mathbf{q}}t+0t} + \int_0^\infty dte^{i\omega t-i\omega_{\mathbf{q}}t-0t}\right] \\
&= -i\frac{\omega_{\mathbf{q}}}{2}\left[\frac{1}{i(\omega+\omega_{\mathbf{q}}-\mathrm{i}0)} + \frac{-1}{i(\omega-\omega_{\mathbf{q}}+\mathrm{i}0)}\right] \\
&= \frac{\omega_{\mathbf{q}}}{2}\left[\frac{1}{\omega-\omega_{\mathbf{q}}+\mathrm{i}0} - \frac{1}{\omega+\omega_{\mathbf{q}}-\mathrm{i}0}\right] \\
&= \frac{\omega_{\mathbf{q}}^2}{\omega^2-\omega_{\mathbf{q}}^2+\mathrm{i}0}.
\end{aligned} \tag{6}$$

Fourier Transform back to $\mathbf{r}$, t:

$$D(\mathbf{q},t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\exp(-\mathrm{i}\omega t)\omega_{\mathbf{q}}^2}{\omega^2-\omega_{\mathbf{q}}^2+\mathrm{i}0} \tag{7}$$

$$= -2\mathrm{i}\pi^2 q^2\kappa\exp\bigl(-\mathrm{i}q^2\kappa|t|\bigr) \tag{8}$$

$$D(\mathbf{r},t) = -2\mathrm{i}\pi^2\kappa\int_\Lambda\frac{\mathrm{d}^2q}{(2\pi)^2}\exp(\mathrm{i}\mathbf{q}\mathbf{r})q^2\exp\bigl(-\mathrm{i}q^2\kappa|t|\bigr) \tag{9}$$

Here Λ on the integral sign reminds us of the cutoff.

$$D(\mathbf{r},t) = -2\mathrm{i}\pi^2\kappa\int_0^\Lambda\frac{\mathrm{d}q}{(2\pi)}q^3\int\frac{\mathrm{d}\phi}{2\pi}\exp(-\mathrm{i}qr\cos(\phi)\kappa|t|) \tag{10}$$

$$= -2\mathrm{i}\pi^2\kappa\int_0^\Lambda\frac{\mathrm{d}q}{(2\pi)}q^3B_0(qr\kappa|t|) \tag{11}$$

where B_0 is the first kind zeroth order Bessel function. we find

$$D(\mathbf{r},t) = -\frac{\mathrm{i}\pi\Lambda^2}{(r\kappa|t|)^2}\left[2B_2(r\kappa|t|\Lambda) - r\kappa|t|\Lambda B_3(r\kappa|t|\Lambda)\right]. \tag{12}$$

More convenient: Introduce exponential cutoff:

$$D(\mathbf{r},t) = -2\mathrm{i}\pi^2\kappa\int_0^\infty\frac{\mathrm{d}q}{(2\pi)}\exp(-q/\Lambda)q^3B_0(qr\kappa|t|) \tag{13}$$

$$= -\mathrm{i}\pi\kappa\frac{6-9\Lambda^2(r\kappa|t|)^2}{\Lambda^3\left(\frac{1}{\Lambda^2}+(r\kappa|t|)^2\right)^{7/2}} \tag{14}$$

2. Polarizability of a particle in a 1D potential (8 + 10 + 12 points)

Consider a charged particle in a one-dimensional system with a potential well characterized by the amplitude V_0 and the spatial range a. Assume that $V_0 \ll \hbar^2/(2ma^2)$. Suppose the particle is in the ground state. The polarizability χ in a weak external electric field $\mathbf{E}$ relates the polarization (dipole moment) with the field: $\mathbf{P} = \chi\mathbf{E}$.

(a) Write down the expression for the Green's function in the momentum representation in the absence of electric field in terms of the scattering amplitude F. Solve the equation for $F(\varepsilon, p_1 \approx 0, p_2 \approx 0)$ when $V_0 \ll \hbar^2/(2ma^2)$ by assuming that F does not have any poles in momentum space. Estimate the bound state energy ϵ_0.

Solution: Let us consider a square well potential for definiteness

$$V(x) = \theta(a/2 - |x|)V_0 \qquad V_0 < 0 \tag{15}$$

$$\Rightarrow V(k) = V_0 \int_{-a/2}^{a/2} \mathrm{d}x \exp(-\mathrm{i}kx) \tag{16}$$

$$= \frac{V_0}{-\mathrm{i}k} [\exp(-\mathrm{i}ak/2) - \exp(\mathrm{i}ak/2)] \tag{17}$$

$$= 2V_0 \frac{\sin(ka/2)}{k}. \tag{18}$$

From now on we set $a = \hbar = 1$.
We consider a particle in a bound state with energy ϵ_0. Bound state implies

$$|\epsilon_0| < |V_0|, \quad \epsilon_0 < 0. \tag{19}$$

We describe the bound state in terms of the retarded Green's function $G^{\mathrm{R}}(\epsilon_0, p_1, p_2)$ (from now on we drop the label "R" in all expressions). As it will turn out, we need the Green's function around $p_1 \approx p_2 \approx 0$ (see task (c)).
From the lectures we know that we can write the Green's function as

$$G = G_0 + G_0 F G_0, \tag{20}$$

where

$$F(\varepsilon, p_1, p_2) = V(p_1 - p_2) + \int \frac{dp_3}{2\pi} V(p_1 - p_3) G_0(\varepsilon, p_3) V(p_3 - p_2) + \ldots. \tag{21}$$

The Green's function in the energy-momentum space can thus be written as

$$G(\varepsilon, p_1, p_2) = \frac{2\pi\delta(p_1 - p_2)}{\varepsilon - \varepsilon_{p_1} + \mathrm{i}0} + \frac{F(\varepsilon, p_1, p_2)}{(\varepsilon - \varepsilon_{p_1} + \mathrm{i}0)(\varepsilon - \varepsilon_{p_2} + \mathrm{i}0)}. \tag{22}$$

The function $F(\varepsilon, p_1, p_2)$ in a 1D system satisfies the equation

$$F(\varepsilon, p_1, p_2) = V(p_1 - p_2) + \int \frac{dp_3}{2\pi} \frac{V(p_1 - p_3) F(\varepsilon, p_3, p_2)}{\varepsilon - p_3^2/(2m) + \mathrm{i}0}. \tag{23}$$

We are interested in evaluating F at the bound state energy $\varepsilon = \epsilon_0 < 0$. Because we assume that V_0 is small, the poles $p_0 = \pm \mathrm{i}\sqrt{2m|\epsilon_0|}$ are close to the real axis, and the integrand is sharply peaked around $p_3 = 0$. On the other hand, when a is small, V varies very slowly. If we choose p_1 and p_2 close to the origin, we can approximate $V(q) \approx V_0$ in the first term and also inside the integral.
Consider this equation around $F(\epsilon_0, 0, 0)$:

$$F(\epsilon_0, 0, 0) = V_0 + \int \frac{\mathrm{d}p_3}{2\pi} \frac{V(-p_3) F(\epsilon_0, p_3, 0)}{\epsilon_0 - p_3^2/(2m) + \mathrm{i}0} \tag{24}$$

where we used $V(k=0) = V_0$.
Assuming, that $F(\epsilon_0, p_3, 0)$ does not have any peaks along p_3 for $p_3 \neq 0$, this integral is determined by the peaks from the denominator, at $p_3 = 0$.
We thus obtain an algebraic equation

$$F(\epsilon_0, 0, 0) = V_0 + V_0 F(\epsilon_0, 0, 0) \int \frac{\mathrm{d}p_3}{2\pi} \frac{1}{\epsilon_0 - p_3^2/(2m) + \mathrm{i}0} \tag{25}$$

$$= V_0 - V_0 F(\epsilon_0, 0, 0) \sqrt{\frac{m}{2|\epsilon_0|}}. \tag{26}$$

From Eq. (23) we see that this equation is stable to perturbations in $p_1 \approx p_2 \approx 0$. We thus find

$$F(\epsilon_0, p_1 \approx 0, p_2 \approx 0) := F(\epsilon_0) \tag{27}$$

$$= \frac{V_0}{1 + V_0 \sqrt{\frac{m}{2|\epsilon_0|}}}. \tag{28}$$

Since $V_0 < 0$, this expression has a pole at

$$1 = |V_0| \sqrt{\frac{m}{2|\epsilon_0|}} \tag{29}$$

$$\Rightarrow \epsilon_0 = -\frac{V_0^2 m}{2}. \tag{30}$$

This pole indicates the existence of a bound state: F is related to the full Green's function through Eq. (20), and the Green's function can be written

$$G(\varepsilon, p_1, p_2) = \sum_\alpha \frac{\psi_\alpha^*(p_1)\psi_\alpha(p_2)}{\varepsilon - \varepsilon_\alpha}. \tag{31}$$

In summary, we find a single bound state in the limit $|V_0| m \ll 1$.

(b) Express the dipole moment $P = \int \mathrm{d}x\, x n(x)$ of the system in terms of the exact Green's function in the momentum representation. You should find

$$P = ie \int \frac{dp_1}{2\pi} \left[\frac{\partial}{\partial p_1} \mathrm{Res} G^R(\varepsilon, p_1, p_2) \right] \Bigg|_{\varepsilon = \epsilon_0 + \mathrm{i}0; p_1 = p_2}.$$

Solution:

$$P = e \int \mathrm{d}x\, x n(x) \tag{32}$$

$$= -\frac{e}{\pi} \int \mathrm{d}x\, x \int_{-\infty}^{0} \mathrm{d}\varepsilon\, \mathrm{Im}(G(\varepsilon, x, x)) \tag{33}$$

$$= e \int \mathrm{d}x\, x |\psi_{\alpha(\epsilon_0)}(x)|^2 \tag{34}$$

In the second equality, we used (see last solution, Eq. (29))

$$\mathrm{Im}\left(G^{\mathrm{R}}(\varepsilon; \mathbf{r}, \mathbf{r})\right) = -\pi \sum_\alpha |\psi_\alpha(\mathbf{r})|^2 \delta\left(\varepsilon - \varepsilon_\alpha\right) \tag{35}$$

and the fact that there is exactly one bound state in the considered limit $V_0 m \to 0$. To rewrite the integrand from Eq. (34) we use

$$|\psi_{\alpha(\epsilon_0)}(x)|^2 = \lim_{\varepsilon\to\epsilon_0} (\varepsilon - \epsilon_0) \left[\sum_\alpha \frac{|\psi_\alpha(x)|^2}{(\varepsilon - \varepsilon_\alpha)}\right] \tag{36}$$

$$= \mathrm{Res}_{\varepsilon\to\epsilon_0} G(\varepsilon, x, x). \tag{37}$$

Thus, we have

$$P(x) = e \int \mathrm{d}x\, x \mathrm{Res}_{\varepsilon\to\epsilon_0} G(\varepsilon, x, x) \tag{38}$$

$$= e \int \frac{\mathrm{d}p_1}{2\pi} \frac{\mathrm{d}p_2}{2\pi} \left[\int \mathrm{d}x\, x \exp(\mathrm{i}x(p_1 - p_2))\right] \mathrm{Res}_{\varepsilon\to\epsilon_0} G(\varepsilon, p_1, p_2) \tag{39}$$

$$= -e\mathrm{i} \int \frac{\mathrm{d}p_1}{2\pi} \frac{\mathrm{d}p_2}{2\pi} \left[2\pi \frac{\partial}{\partial p_1} \delta(p_1 - p_2)\right] \mathrm{Res}_{\varepsilon\to\epsilon_0} G(\varepsilon, p_1, p_2) \tag{40}$$

$$= e\mathrm{i} \int \frac{\mathrm{d}p_1}{2\pi} \frac{\mathrm{d}p_2}{2\pi} (2\pi\delta(p_1 - p_2)) \frac{\partial}{\partial p_1} \mathrm{Res}_{\varepsilon\to\epsilon_0} G(\varepsilon, p_1, p_2) \tag{41}$$

$$= e\mathrm{i} \int \frac{\mathrm{d}p_1}{2\pi} \left[\frac{\partial}{\partial p_1} \mathrm{Res}_{\varepsilon\to\epsilon_0} G(\varepsilon, p_1, p_2)\right]_{p_2=p_1}. \tag{42}$$

(c) Consider now the Green's function in the potential

$$W = -eEx$$

induced by the applied electric field. What is a graphical representation for the linear-in-E correction to the Green's function? Using the diagrams, evaluate the polarizability of the system.

Hint: Why can you use F with $p_1 \approx 0$, $p_2 \approx 0$?

Solution:

A weak electric field is applied to the system. The Green's function of the system including the electric field is defined by the equation

$$(\varepsilon - H - W) G_W(\varepsilon) = \mathbb{1} \tag{43}$$

where $H = H_0 + V$. We can use the Green's function of H (calculated in the first subtask) to find G_W:

$$G_W(\varepsilon) = \frac{\mathbb{1}}{G^{-1}(\varepsilon) - W} \tag{44}$$

$$= G(\varepsilon) \sum_{n=0}^{\infty} [W G(\varepsilon)]^n. \tag{45}$$

We can also do this without operator notation explicitly:

$$D_x := \varepsilon - H_x \tag{46}$$

$$D_x G_{x,x'}(\varepsilon) = \delta_{x,x'} \tag{47}$$

$$(D_x + W_x) G^W_{x,x'}(\varepsilon) = \delta_{x,x'} \tag{48}$$

$$\Rightarrow D_x G^W_{x,x'}(\varepsilon) = \delta_{x,x'} - W_x G^W_{x,x'}(\varepsilon) \tag{49}$$

Using G to solve the last equation (dropping energy argument in the notation):

$$G^W_{x,x'} = \int \mathrm{d}x''\, G_{x,x''}[\delta_{x'',x'} + W_{x''}G_{x'',x'}] \tag{50}$$

$$= G_{x,x'} + \int \mathrm{d}x''\, W_{x''}G^W_{x'',x'} \tag{51}$$

Fourier transformation of second term:

$$\int \mathrm{d}x'' \frac{\mathrm{d}p}{2\pi}\frac{\mathrm{d}p'}{2\pi} \mathrm{e}^{\mathrm{i}px}\mathrm{e}^{-\mathrm{i}p'x'} G_{x,x''}W_{x''}G^W_{x'',x'} = \int \mathrm{d}x''\, G_{p,x''}W_{x''}G^W_{x'',p'} \tag{52}$$

$$= \int \mathrm{d}x'' \int \frac{\mathrm{d}p''}{2\pi} \exp(\mathrm{i}p''x'')G_{p,p''}W_{x''}G^W_{x'',p'} \tag{53}$$

$$= \frac{1}{2\pi}\int \mathrm{d}p''\,\mathrm{d}p'''\, G_{p,p''}W(p''-p''')G^W_{p''',p'} \tag{54}$$

Thus:

$$G_W(\varepsilon, p_1, p_2) = G(\varepsilon, p_1, p_2) + \frac{1}{2\pi}\int \mathrm{d}p_3\,\mathrm{d}p_4\, G(\varepsilon, p_1, p_3)W(p_3-p_4)G_W(\varepsilon, p_4, p_2) \tag{55}$$

Up to linear order in W we find

$$G_W(\varepsilon) = G(\varepsilon) + G(\varepsilon)WG(\varepsilon). \tag{56}$$

At this point, we note that it is justified to use the expression for $F(\epsilon_0, p_1 \approx 0, p_2 \approx 0)$ in this formula. Due to Eq. (42) we evaluate the Green's function at $\varepsilon = \epsilon_0$. Again, in the given limit $|V_0|m \ll 1$ the Green's functions $G_0(\epsilon_0, p)$ are strongly peaked around $p = 0$, allowing to substitute the slowly changing $F(\epsilon_0, p_1, p_2)$ by its value in the peak region.
To determine the polarizability

$$\chi = |P/E|, \tag{57}$$

we plug in the term induced by the electric field

$$\delta G(\varepsilon) = G_W(\varepsilon) - G(\varepsilon) \tag{58}$$

$$= G(\varepsilon)WG(\varepsilon) \tag{59}$$

into the expression for the dipole moment (42).
Expressing G through G_0 and F, we have

$$\delta G = G_0WG_0 + G_0FG_0WG_0 + G_0WG_0FG_0 + G_0FG_0WG_0FG_0 \tag{60}$$

$$:= \delta G^{(1)} + \delta G^{(2)} + \delta G^{(3)} + \delta G^{(4)}. \tag{61}$$

The momentum space representation of the electric field reads

$$W(p) = -eE\int \mathrm{d}x \exp(-\mathrm{i}px)x \tag{62}$$

$$= -2\pi\mathrm{i}eE\frac{\partial}{\partial p}\delta(p). \tag{63}$$

We calculate the terms one by one:

$$\delta G^{(1)}(\varepsilon, p_1, p_2) = \frac{1}{2\pi} \int \mathrm{d}p_3\, \mathrm{d}p_4\, G_0(\varepsilon, p_1, p_3) W(p_3 - p_4) G_0(\varepsilon, p_4, p_2) \tag{64}$$

$$= -\mathrm{i}eE \int \mathrm{d}p_3\, \mathrm{d}p_4\, \delta'(p_3 - p_4) \frac{\delta(p_1 - p_3)}{\varepsilon - p_1^2/(2m) + \mathrm{i}0} \frac{\delta(p_4 - p_2)}{\varepsilon - p_4^2/(2m) + \mathrm{i}0} \tag{65}$$

$$= -\mathrm{i}eE \int \mathrm{d}p_4\, \frac{\delta'(p_1 - p_4)}{\varepsilon - p_1^2/(2m) + \mathrm{i}0} \frac{\delta(p_4 - p_2)}{\varepsilon - p_4^2/(2m) + \mathrm{i}0} \tag{66}$$

$$= \mathrm{i}eE \frac{p_1}{(\varepsilon - p_1^2/(2m) + \mathrm{i}0)^2} \frac{\delta(p_1 - p_2)}{\varepsilon - p_2^2/(2m) + \mathrm{i}0} \tag{67}$$

This can be seen to vanish when plugged into the expression for P (Symmetric integration interval, antisymmetric integrand):

$$\delta G^{(1)}(\varepsilon, x, x) \propto \int \mathrm{d}p_1\, \mathrm{d}p_2 \exp(\mathrm{i}x(p_1 - p_2)) \frac{p_1}{(\varepsilon - p_1^2/(2m) + \mathrm{i}0)^2} \frac{\delta(p_1 - p_2)}{\varepsilon - p_2^2/(2m) + \mathrm{i}0} \tag{68}$$

$$= \int \mathrm{d}p_1\, \frac{p_1}{(\varepsilon - p_1^2/(2m) + \mathrm{i}0)^2} \frac{1}{\varepsilon - p_1^2/(2m) + \mathrm{i}0} \tag{69}$$

therefore, it does not contribute to P (see Eq. (38)).
Second term:

$$\delta G^{(2)}(\varepsilon, p_1, p_2) = \frac{1}{2\pi} \int \mathrm{d}p_3\, \mathrm{d}p_4\, G_0(\varepsilon, p_1, p_3) W(p_3 - p_4) G_0(\varepsilon, p_4) F(\varepsilon) G_0(\varepsilon, p_2) \tag{70}$$

$$= \frac{1}{2\pi} \int \mathrm{d}p_4\, G_0(p_1) W(p_1 - p_4) G_0(p_4) F(\varepsilon) G_0(p_2) \tag{71}$$

$$= -\mathrm{i}eEF(\varepsilon) \int \mathrm{d}p_4\, G_0(\varepsilon, p_1) \delta'(p_1 - p_4) G_0(\varepsilon, p_4) G_0(\varepsilon, p_2) \tag{72}$$

$$= -\mathrm{i}eEF(\varepsilon) G_0(\varepsilon, p_1) G_0'(\varepsilon, p_1) G_0(\varepsilon, p_2) \tag{73}$$

$$= -\frac{\mathrm{i}eEF(\varepsilon)}{m} \frac{p_1}{(\varepsilon - p_1^2/(2m) + \mathrm{i}0)^3} \frac{1}{\varepsilon - p_2^2/(2m) + \mathrm{i}0} \tag{74}$$

Similarly, the third term:

$$\delta G^{(3)}(\varepsilon, p_1, p_2) = F(\varepsilon) \frac{1}{2\pi} \int \mathrm{d}p_3\, \mathrm{d}p_4\, G_0(\varepsilon, p_1) G_0(\varepsilon, p_3) W(p_3 - p_4) G_0(\varepsilon, p_4) \delta(p_2 - p_4) \tag{75}$$

$$= -\mathrm{i}eEF(\varepsilon) \int \mathrm{d}p_3\, G_0(\varepsilon, p_1) G_0(\varepsilon, p_3) \delta'(p_3 - p_2) G_0(\varepsilon, p_2) \tag{76}$$

$$= \frac{\mathrm{i}eEF(\varepsilon)}{m} \frac{1}{\varepsilon - p_1^2/(2m) + \mathrm{i}0} \frac{p_2}{(\varepsilon - p_2^2/(2m) + \mathrm{i}0)^3} \tag{77}$$

Last term:

$$\delta G^{(4)}(\varepsilon, p_1, p_2) = F(\varepsilon)^2 \frac{1}{2\pi} \int \mathrm{d}p_3\, \mathrm{d}p_4\, G_0(\varepsilon, p_1) G_0(\varepsilon, p_3) W(p_3 - p_4) G_0(\varepsilon, p_4) G_0(\varepsilon, p_2) \tag{78}$$

$$= -\mathrm{i}eEF(\varepsilon)^2 G_0(\varepsilon, p_1) G_0(\varepsilon, p_2) \int \mathrm{d}p_4\, G_0'(\varepsilon, p_3) G_0(\varepsilon, p_3) \tag{79}$$

$$= -\mathrm{i}eEF(\varepsilon)^2 G_0(\varepsilon, p_1) G_0(\varepsilon, p_2) \int \mathrm{d}p_3\, \frac{p_3}{(\varepsilon - p_3^2/(2m) + \mathrm{i}0)^3} \tag{80}$$

$$= 0. \tag{81}$$

(Integration of antisymmetric function over symmetric interval.)
We end up with

$$\delta G(\varepsilon, p_1, p_2) = \frac{\mathrm{i}eEF(\varepsilon)}{m} G_0(\varepsilon, p_1) G_0(\varepsilon, p_2) \left[p_2 G_0^2(\varepsilon, p_2) - p_1 G_0^2(\varepsilon, p_1) \right] \tag{82}$$

To find P, we calculate the residue of this expression (see Eq. (42))

$$\lim_{\varepsilon \to \epsilon_0} [\varepsilon - \epsilon_0] \delta G(\varepsilon, p_1, p_2) = \frac{\mathrm{i}eE}{m} G_0(\epsilon_0, p_1) G_0(\epsilon_0, p_2) \left[p_2 G_0^2(\epsilon_0, p_2) - p_1 G_0^2(\epsilon_0, p_1) \right] \lim_{\varepsilon \to \epsilon_0} [\varepsilon - \epsilon_0] F(\varepsilon) \tag{83}$$

Isolating the pole in $F(\varepsilon)$ at $\varepsilon = \epsilon_0$:

$$F(\varepsilon) = -\frac{|V_0|}{1 - |V_0| \sqrt{\frac{m}{2|\varepsilon|}}} \tag{84}$$

$$= -\frac{|V_0|(1 + |V_0| \sqrt{\frac{m}{2|\varepsilon|}})}{1 - \frac{V_0^2 m}{2|\varepsilon|}} \tag{85}$$

$$= -|\varepsilon| \frac{|V_0|(1 + |V_0| \sqrt{\frac{m}{2|\varepsilon|}})}{|\varepsilon| - \frac{V_0^2}{2m}} \tag{86}$$

Therefore:

$$\Rightarrow \lim_{\varepsilon \to \epsilon_0} [\varepsilon - \epsilon_0] F(\varepsilon) = m|V_0|^3 \tag{87}$$

$$\lim_{\varepsilon \to \epsilon_0} \delta G(\varepsilon, p_1, p_2) = \mathrm{i}eE|V_0|^3 G_0(\epsilon_0, p_2) \left[p_2 G_0^2(\epsilon_0, p_2) - p_1 G_0^2(\epsilon_0, p_1) \right] \tag{88}$$

And using mathematica to evaluate this in the epxression for P (42)

$$P = \mathrm{i}e \int \frac{\mathrm{d}p_1}{2\pi} \left[\frac{\partial}{\partial p_1} \lim_{\varepsilon \to \epsilon_0} \delta G(\varepsilon, p_1, p_2) \right]_{p_2 = p_1} = -\frac{5e^2 E}{16\epsilon_0^2 m} \tag{89}$$

Finally

$$\chi = |P/E| = \frac{5e^2}{16\epsilon_0^2 m} \tag{90}$$

The polarizability captures, how strongly the dipole moment (32) of the system changes due to the application of a weak electric field. From our result we read off that the change in the dipole moment increases as the binding energy tends to zero. This is expected, since a weakly bound particle can be displaced more easily.