Karlsruher Institut für Technologie   Institute for Condensed Matter Theory
Institute for Quantum Materials and Technologies

Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 3
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1. Holstein–Primakoff transformation (15 points)

The Holstein–Primakoff transformation, defined as

$$\hat{S}_+ = \hbar\sqrt{2S}\sqrt{1-\frac{\hat{b}^\dagger\hat{b}}{2S}}\; b\,, \qquad \hat{S}_- = \hbar\sqrt{2S}\,\hat{b}^\dagger\sqrt{1-\frac{\hat{b}^\dagger\hat{b}}{2S}}\,, \qquad \hat{S}_z = \hbar\left(S-\hat{b}^\dagger\hat{b}\right)\,,$$

expresses the spin operators $\hat{S}_+$, $\hat{S}_-$, and $\hat{S}_z$ for a spin S through bosonic creation and annihilation operators $\hat{b}^\dagger$ and $\hat{b}$. This transformation is particularly useful when $S \gg 1$: in this case, the square roots can be expanded in Taylor series of powers of $1/S$.

Demonstrate that the operators defined above indeed obey the commutation relations for spin-S operators $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$, $\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-)$ and S_z:

$$\left[\hat{S}_x, \hat{S}_x\right] = 0, \quad \left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z, \quad \left[\hat{S}_x, \hat{S}_z\right] = -i\hbar\hat{S}_y, \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x.$$

2. Wick's theorem: ($10 + 10$ points)

The Wick theorem states that a time-ordered product of operators can be rewritten as the normal-ordered product of these operators plus the normal-ordered products with all single contractions among operators plus the normal-ordered products with all double contractions, etc., plus all full contractions (see lecture notes, Sec. 3.8.1).

(a) In the lectures, we assumed that the operators entering the time-ordered (chronological) product are linear in creation/annihilation operators. Is this assumption necessary for the validity of Wick's theorem? Why?

(b) Would Wick's theorem be valid if we replaced the chronological product in the theorem, as well as in the definition of contraction, by a product that orders the operators according to their coordinate along the x-axis? Substantiate your answer.

3. Spectral weight (15 points)

Demonstrate that the spectral weight in a many-body fermionic system is normalized:

$$\int d\varepsilon \mathcal{A}(\mathbf{p}, \varepsilon) = 1.$$

Determine the leading asymptotics of the fermionic Green's functions in the energy-momentum representation for $\varepsilon \to \infty$.