Karlsruher Institut für Technologie Institute for Condensed Matter Theory
Institute for Quantum Materials and Technologies

**Condensed Matter Theory II: Many-Body Theory (TKM II) SoSe 2023**

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**Homework assignment 5**
**Deadline: 26 May 2023**

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**1. Hartree-Fock self-energy** $(10 + 7 + 3$ points$)$

Consider a three-dimensional system of interacting electrons at zero temperature. Derive carefully the following results, which are given in Sec. (3.12.2) of the lecture notes.

(a) Calculate the Hartree and Fock self-energy for electrons interacting via Coulomb interaction $U(\mathbf{r}, \mathbf{r}') = \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}$.

(b) Calculate the same diagrams with screened Thomas-Fermi interaction $U(\mathbf{r}, \mathbf{r}') = \frac{e^2}{|\mathbf{r}-\mathbf{r}'|} \exp(-\kappa|\mathbf{r} - \mathbf{r}'|)$, where $\kappa^{-1}$ is the screening length.

(c) Calculate the effective mass

$$m^* = m \frac{1 - \frac{\partial}{\partial \epsilon} \mathrm{Re}\, \Sigma(\epsilon, \mathbf{p})\Big|_{\epsilon=0,\, p=p_F}}{1 + \frac{\partial}{\partial \epsilon_p} \mathrm{Re}\, \Sigma(\epsilon, \mathbf{p})\Big|_{\epsilon=0,\, p=p_F}}, \tag{1}$$

for the Thomas-Fermi interaction. What happens at the limit of unscreened interaction, $\kappa \to \infty$?

**2. Quasiparticle life-time in two dimensions** $(3 + 2 + 10$ points$)$

Consider a contact interaction $U(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$ for spinful 2D electrons.

(a) Write down all the self-energy diagrams to the second order in the interaction. Which diagrams contribute to the imaginary part of the self-energy?

(b) Write the expression for the imaginary part of the self-energy. What changes relative to the 3D case, given by Eq. (3.271) in the lectures?

(c) Estimate the quasi-particle lifetime. The result should be Eq. (3.276), but you do not need to calculate the prefactor.

**3. Perturbation expansion for Fermi liquid interaction** (15 points)

In Fermi liquid theory, the total energy of an excited state relative to the ground state energy is given by

$$E - E_0 = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma} + \sum_{\mathbf{p}\sigma, \mathbf{p}'\sigma} f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} \delta n_{\mathbf{p}\sigma} \delta n_{\mathbf{p}'\sigma'}, \tag{2}$$

where $\varepsilon_{\mathbf{p}\sigma}$ is the (renormalized) quasiparticle energy, $\delta n_{\mathbf{p}\sigma} = n_{\mathbf{p}\sigma} - n^{(0)}_{\mathbf{p}\sigma}$ is the difference between the occupation number $n_{\mathbf{p}\sigma}$ in an excited state and the ground state occupation number $n^{(0)}_{\mathbf{p}\sigma}$, and $f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'}$ is the effective Fermi liquid interaction. The interaction can be divided into spin-dependent and spin-independent parts as

$$f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} = f^s_{\mathbf{p}, \mathbf{p}'} + f^a_{\mathbf{p}, \mathbf{p}'} \sigma\sigma'. \tag{3}$$

Consider the Hamiltonian

$$\hat{H} = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma} \hat{n}_{\mathbf{p}\sigma} + \frac{1}{2} \sum_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} V(|\mathbf{q}|) c^\dagger_{\mathbf{p}-\mathbf{q},\sigma} c^\dagger_{\mathbf{p}'+\mathbf{q},\sigma'} c_{\mathbf{p}'\sigma'} c_{\mathbf{p}\sigma}. \tag{4}$$

To the first order in the perturbation theory, calculate the energy of the state

$$|\Psi\rangle = |n_{\mathbf{p}_1\sigma_1}, n_{\mathbf{p}_2\sigma_2}, \ldots\rangle \tag{5}$$

which is some excited state with $n_{\mathbf{p}\sigma}$ electrons in the single-particle states $(\mathbf{p}, \sigma)$. Compare the energy of this state to the noninteracting ground state. Identify the microscopic equivalents of the Landau interaction parameters $f^s$ and $f^a$ to the first order in the interaction. (*Hint: How to calculate the energy in the first order perturbation theory, given the interaction $\hat{V}$?*)