Karlsruher Institut für Technologie Institute for Condensed Matter Theory
Institute for Quantum Materials and Technologies

Condensed Matter Theory II: Many-Body Theory (TKM II) SoSe 2023

PD Dr. I. Gornyi and Prof. Dr. A. Mirlin
Dr. Risto Ojajärvi and Paul Pöpperl

Homework assignment 5
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1. Hartree-Fock self-energy (10 + 7 + 3 points)

Consider a three-dimensional system of interacting electrons at zero temperature. Derive carefully the following results, which are given in Sec. (3.12.2) of the lecture notes.

(a) Calculate the Hartree and Fock self-energy for electrons interacting via Coulomb interaction $U(\mathbf{r}, \mathbf{r}') = \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}$.
Solution: The Hartree and Fock self-energy diagrams are shown in Figure 1.

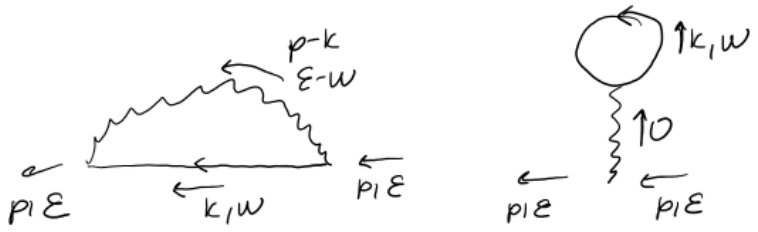

Abbildung 1: First-order self-energy diagrams diagrams

Following the diagrammatic rules, we get the expression ($\eta \to +0$)

$$\begin{aligned}
\Sigma_1(\mathbf{p}, \epsilon) &= i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{d}\omega}{2\pi} \left[(-1)2U(0) + U(\mathbf{p}-\mathbf{k})\right] G(\mathbf{k}, \omega) e^{i\eta\omega} \\
&= -i \int \frac{\mathrm{d}^3q}{(2\pi)^3} \left[2U(0) - U(\mathbf{p}-\mathbf{k})\right] \int \frac{\mathrm{d}\omega}{2\pi} \frac{e^{i\eta\omega}}{\omega - \epsilon_k + i0\mathrm{sign}(\epsilon_k)} \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[2U(0) - U(\mathbf{p}-\mathbf{k})\right] \underbrace{n_k}_{\Theta(p_F-k)} \qquad (1) \\
&= \frac{1}{(2\pi)^2} \int_0^{p_F} \mathrm{d}k k^2 \int_{-1}^{1} \mathrm{d}\cos\phi \left[2U(0) - U(\mathbf{p}-\mathbf{k})\right]. \qquad (2)
\end{aligned}$$

Here, θ is the angle between vectors $\mathbf{p}$ and $\mathbf{k}$ (the self-energy does not depend on the direction of $\mathbf{p}$). The Fourier transform of the Coulomb potential is given by

$$U(\mathbf{q}) = \frac{4\pi e^2}{q^2}. \qquad (3)$$

One immediately sees that the Hartree term diverges, as it contains $U(0)$ (see discussion in the Lecture Notes: this term is, in fact, cancelled by the positive background). However, this term does not depend on ϵ and $\mathbf{p}$, and hence it does not contribute to the effective mass.

We now evaluate the second, Fock term with $-U(\mathbf{p}-\mathbf{k})$ in the integrand. Assuming for definiteness $p > p_F$, we find:

$$\Sigma_F(p,\epsilon) = -\frac{4\pi e^2}{(2\pi)^2}\int_0^{p_F} \mathrm{d}k \int_{-1}^{1} \mathrm{d}\cos\phi \frac{k^2}{p^2+k^2-2pk\cos\phi}$$
$$= -\frac{4\pi e^2}{(2\pi)^2}\int_0^{p_F} \mathrm{d}k\, \frac{k}{p}\, \ln\frac{p+k}{p-k} = -\frac{e^2}{\pi}\left(p_F - \frac{p^2-p_F^2}{2p}\ln\frac{p+p_F}{p-p_F}\right). \qquad (4)$$

(b) Calculate the same diagrams with screened Thomas-Fermi interaction $U(\mathbf{r},\mathbf{r}') = \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\exp(-\kappa|\mathbf{r}-\mathbf{r}'|)$, where κ^{-1} is the screening length.
Solution: If we consider the static Thomas-Fermi screening of the Coulomb interaction,

$$U(\mathbf{r}-\mathbf{r}') = \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}e^{-\kappa|\mathbf{r}-\mathbf{r}'|}, \qquad U(\mathbf{q}) = \frac{4\pi e^2}{q^2+\kappa^2}, \qquad (5)$$

the singularity of $U(\mathbf{q})$ at $q = 0$ will be cured at the inverse screening length κ. Electrons that are far apart do not experience the bare Coulomb interaction, because other electrons screen the interaction and reduce its range. Performing the calculation with the Thomas-Fermi screened Coulomb interaction, we get

$$\Sigma_F(p,\epsilon) = -\frac{4\pi e^2}{(2\pi)^2}\int_0^{p_F} \mathrm{d}k\, \frac{k}{2p}\, \ln\frac{(p+k)^2+\kappa^2}{(p-k)^2+\kappa^2} \qquad (6)$$
$$= -\frac{e^2}{\pi}\Bigg[\Big(\frac{\kappa^2-p^2}{2p}\tanh^{-1}\left(\frac{2pp_F}{\kappa^2+p^2+p_F^2}\right) + \frac{p_F^2}{4p}\log\left(\frac{4pp_F}{\kappa^2+(p-p_F)^2}+1\right)$$
$$+ \left(\kappa\tan^{-1}\left(\frac{p-p_F}{\kappa}\right) - \kappa\tan^{-1}\left(\frac{p+p_F}{\kappa}\right) + p_F\right)\Bigg]$$

This can also be expressed in terms of logs as the unscreened result.
Remarks: The result (14) is controllable for $\kappa/p_F \ll 1$, otherwise additional terms should be included. Note that the value of the self-energy at $p = p_F$,

$$\Sigma_F(p_F,0) = -\frac{e^2}{\pi}\left(\frac{\kappa^2\log\left(\frac{4p_F^2}{\kappa^2}+1\right)}{4p_F} - \kappa\tan^{-1}\left(\frac{2p_F}{\kappa}\right) + p_F\right) \qquad (7)$$
$$\simeq -\frac{e^2}{\pi}p_F\left(1-\frac{\pi\kappa}{2p_F}\right), \qquad \kappa \ll p_F \qquad (8)$$

renormalizes the chemical potential $\mu^* = \mu + \Sigma(p_F,0)/Z$. Note also that with the static RPA screening, the self-energy remains ϵ-independent, so that there are still no corrections to the quasiparticle residue: $Z = 1$.

(c) Calculate the effective mass

$$m^* = m\frac{1-\frac{\partial}{\partial\epsilon}\mathrm{Re}\,\Sigma(\epsilon,\mathbf{p})\big|_{\epsilon=0,\,p=p_F}}{1+\frac{\partial}{\partial\epsilon_p}\mathrm{Re}\,\Sigma(\epsilon,\mathbf{p})\Big|_{\epsilon=0,\,p=p_F}}, \qquad (9)$$

for the Thomas-Fermi interaction. What happens at the limit of unscreened interaction, $\kappa \to \infty$?

Solution:
Our self-energies do not depend on energy and are real. The effective mass simplifies to

$$m^* = \frac{m}{1 + \frac{\partial}{\partial \epsilon_p}\Sigma(\epsilon, \mathbf{p})\Big|_{\epsilon=0,\, p=p_F}}, \tag{10}$$

The numerator was related to quasiparticle weight, which is not affected by these diagrams:

$$Z^{(1)} = \frac{1}{1 - \frac{\partial}{\partial \epsilon}\mathrm{Re}\,\Sigma_1(\mathbf{p}, \epsilon)\Big|_{\epsilon=0,\, p=p_F}} = 1. \tag{11}$$

For unscreened Coulomb interaction, the expression in the denominator of Eq. (9),

$$\begin{aligned}\frac{\partial \Sigma_1(p,\epsilon)}{\partial \epsilon_p} &= -\frac{m}{p}\frac{4\pi e^2}{(2\pi)^2}\frac{\partial}{\partial p}\left(p_F - \frac{p^2 - p_F^2}{2p}\ln\frac{p+p_F}{p-p_F}\right) \\ &= \frac{e^2 m}{\pi p}\left(\frac{p^2+p_F^2}{2p^2}\ln\frac{p+p_F}{p-p_F} - \frac{p_F}{p}\right) \underbrace{\to}_{p\to p_F} \frac{e^2 m}{\pi p_F}\ln\frac{2p_F}{p-p_F},\end{aligned} \tag{12}$$

diverges logarithmically at $p = p_F$. Thus, to 1st order in the Coulomb interaction,

$$m^* = 0. \tag{13}$$

This result is unphysical and is cured by higher order diagrams (RPA sum).
For Thomas-Fermi:

$$\begin{aligned}\frac{\partial \Sigma_F(p,\epsilon)}{\partial \epsilon_p} &= \frac{e^2 m}{\pi p}\left(\frac{\kappa^2 + p_F^2 + p^2}{4p^2}\ln\frac{\kappa^2 + (p+p_F)^2}{\kappa^2 + (p-p_F)^2} - \frac{p_F}{p}\right), \\ \frac{\partial \Sigma_F(p,\epsilon)}{\partial \epsilon_p}\bigg|_{p=p_F} &= \frac{e^2 m}{\pi p_F}\left(\frac{\kappa^2 + 2p_F^2}{4p_F^2}\ln\frac{\kappa^2 + 4p_F^2}{\kappa^2} - 1\right).\end{aligned}$$

These diagrams are real. Using $\kappa^2 = 4\pi e^2 \nu$ with the density of states at the Fermi level $\nu = m p_F/\pi^2$, we obtain

$$m^* = \frac{m}{1 + \frac{\kappa^2}{4p_F^2}\left(\frac{\kappa^2+2p_F^2}{4p_F^2}\ln\frac{\kappa^2+4p_F^2}{\kappa^2} - 1\right)} \underbrace{\to}_{\kappa \ll p_F} m\left(1 - \frac{\kappa^2}{4p_F^2}\ln\frac{2p_F}{\kappa}\right), \tag{14}$$

in which the logarithm no longer diverges and the mass stays finite.

2. Quasiparticle life-time in two dimensions (3 + 2 + 10 points)

Consider a contact interaction $U(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$ for spinful 2D electrons.

(a) Write down all the self-energy diagrams to the second order in the interaction. Which diagrams contribute to the imaginary part of the self-energy?
Solution: According to the lecture notes only the $\Sigma^{(2b1)}$ and $\Sigma^{(2b2)}$ diagrams contribute, since the other diagrams are purely real. The $\Sigma^{(2b1)}$ diagram we will calculate below. One can see that in case of the contact interaction, $\Sigma^{(2b1)} = -2\Sigma^{(2b1)}$:

$$\Sigma^{(2b1)}(\mathbf{p},\epsilon) = (-1)\sum_\sigma i^2 g^2 \sum_{\mathbf{k},\mathbf{q},\epsilon',\nu} G(\mathbf{p}+\mathbf{q}, \epsilon+\nu)G(\mathbf{k}+\mathbf{q},\epsilon'+\nu)G(\mathbf{k},\epsilon') \tag{15}$$

$$\Sigma^{(2b1)}(\mathbf{p},\epsilon) = \qquad i^2 g^2 \sum_{\mathbf{k},\mathbf{q},\epsilon',\nu} G(\mathbf{p}+\mathbf{q}, \epsilon+\nu)G(\mathbf{k}+\mathbf{q},\epsilon'+\nu)G(\mathbf{k},\epsilon') \tag{16}$$

(b) Write the expression for the imaginary part of the self-energy. What changes relative to the 3D case, given by Eq. (3.271) in the lectures?

Solution: Only the momentum integration changes,

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \to \int \frac{\mathrm{d}^2 p}{(2\pi)^2}. \tag{17}$$

(c) Estimate the quasi-particle lifetime. The result should be Eq. (3.276), but you do not need to calculate the prefactor.

Solution:

Above, we found that we need to calculate

$$\begin{aligned}\operatorname{Im}\Sigma(\mathbf{p},\epsilon_{\mathbf{p}}) = -\frac{g^2\pi}{(2\pi)^4}\int \mathrm{d}^2 q \Theta(|\mathbf{p}+\mathbf{q}|-p_F) \\ \times \int \mathrm{d}^2 k \delta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{k}+\mathbf{q}})\Theta(p_F-|\mathbf{k}+\mathbf{q}|)\Theta(|\mathbf{k}|-p_F).\end{aligned} \tag{18}$$

The integral can only be nonzero when $|\mathbf{p}|, |\mathbf{p}+\mathbf{q}|, |\mathbf{k}| > p_F$ and $|\mathbf{k}+\mathbf{q}| < p_F$. From energy conservation we also find that

$$\varepsilon_{\mathbf{p}} = \varepsilon_{\mathbf{p}+\mathbf{q}} + (\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}}) > \varepsilon_{\mathbf{p}+\mathbf{q}}, \tag{19}$$

the energy of the scattered particle is less that the original energy, and $p > k > p_F$.

Let us draw the allowed values of $\mathbf{q}$ on x-y plane. Because $\mathbf{k}+\mathbf{q}$ is a hole, the values of $\mathbf{q}$ within a ball of radius p_F centered at (k_x, k_y) are allowed. Because $\mathbf{p}+\mathbf{q}$ is an electron and $|\mathbf{p}+\mathbf{q}| < p$, $\mathbf{q}$ has to be inside a spherical shell of inner radius p_F and outer radius p centered at (p_x, p_y). Now, if $\mathbf{p}$ and $\mathbf{k}$ point to opposite directions, these two regions do not overlap and the integral over $\mathbf{q}$ vanishes. The allowed phase space is maximized for forward scattering in which $\mathbf{p}$ and $\mathbf{k}$ point to the same direction, and the transferred momentum $\mathbf{q}$ is small. This does not immediately give us a phase space argument which would allow us to determine the lifetime, but helps to analyze the following evaluation of the integral.

We start with evaluating the $\mathbf{k}$ integral and choose to do it in Cartesian coordinates. For this integral, we choose k_x to point along the direction of $\mathbf{q}$ and k_y along the perpendicular direction. Then we get on the mass-shell ($\varepsilon = \varepsilon_{\mathbf{p}}$) that

$$\begin{aligned}\delta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{k}+\mathbf{q}}) &= \delta\left(\frac{\mathbf{q}\cdot(\mathbf{k}-\mathbf{p})}{2m}\right) \\ &= \delta\left(\frac{q}{m}(k_x - p\cos\phi)\right) \\ &= \frac{m}{q}\delta\left(k_x - p\cos\phi\right),\end{aligned} \tag{20}$$

where ϕ is the angle between $\mathbf{q}$ and $\mathbf{p}$. This fixes the projections of $\mathbf{k}$ and $\mathbf{p}$ along $\mathbf{q}$ to be equal. From the theta-functions we get the following integration limits for k_y:

$$\begin{aligned}\Theta(p_F-|\mathbf{k}+\mathbf{q}|)\Theta(|\mathbf{k}|-p_F) &= \Theta(p_F^2-|\mathbf{k}+\mathbf{q}|^2)\Theta(|\mathbf{k}|^2-p_F^2) \\ &= \Theta(p_F-|\mathbf{k}+\mathbf{q}|)\Theta(|\mathbf{k}|-p_F) \\ &= \Theta(p_F^2-(k_x+q)^2-k_y^2)\Theta(k_y^2+k_x^2-p_F^2)\end{aligned} \tag{21}$$

In other words

$$p_F^2 - (k_x + q)^2 > k_y^2 > p_F^2 - k_x^2, \tag{22}$$

This can only be satisfied if $k_x < -q/2$.
Let us not do the k_x integration yet. Instead consider the θ-functions on k_x–k_y plane. The first one, $\Theta(k_y^2 + (k_x - q)^2 - p_F^2)$ tells us that we only integrate over $\mathbf{k}$ inside a ball of radius p_F centered at $(-q, 0)$. The other one tells us to only integrate over the $\mathbf{k}$ that are outside a ball of radius p_F centered at $(0, 0)$. There are two cases: (i) When $2p_F < q$ the balls do not overlap and we only need to consider the first θ, the other one is satisfied automatically. (ii) When $0 < q < 2p_F$, the balls overlap and we integrate over the first ball minus the overlap with the second ball. This gives us the integration regions

$$\begin{aligned}
&\iint \mathrm{d}k_x \mathrm{d}k_y \delta(k_x - p\cos\phi)\Theta(p_F^2 - (k_x+q)^2 - k_y^2)\Theta(k_y^2 + k_x^2 - p_F^2) \\
&= \Theta(q - 2p_F) \int_{-q-p_F}^{-q+p_F} \mathrm{d}k_x \delta(k_x - p\cos\phi) \int_{-\sqrt{p_F^2-(k_x+q)^2}}^{+\sqrt{p_F^2-(k_x+q)^2}} \mathrm{d}k_y && (23) \\
&+ \Theta(2p_F - q) \int_{-q-p_F}^{-p_F} \mathrm{d}k_x \delta(k_x - p\cos\phi) \int_{-\sqrt{p_F^2-(k_x+q)^2}}^{+\sqrt{p_F^2-(k_x+q)^2}} \mathrm{d}k_y && (24) \\
&+ \Theta(2p_F - q) \int_{-p_F}^{-q/2} \mathrm{d}k_x \delta(k_x - p\cos\phi) \left[\int_{-\sqrt{p_F^2-(k_x+q)^2}}^{+\sqrt{p_F^2-(k_x+q)^2}} \mathrm{d}k_y - \int_{-\sqrt{p_F^2-k_x^2}}^{+\sqrt{p_F^2-k_x^2}} \mathrm{d}k_y \right]. && (25)
\end{aligned}$$

Using the energy conservation and the theta functions, we can show that

$$\varepsilon_{\mathbf{p}} = \varepsilon_{\mathbf{p+q}} + (\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k+q}}) > \varepsilon_{\mathbf{k+q}} \implies p > |\mathbf{p} + \mathbf{q}| > p_F \tag{26}$$

From this we find the constraints on $\cos\theta$:

$$\begin{aligned}
|p| > |\mathbf{p} + \mathbf{q}| > p_F &\Leftrightarrow p^2 > p^2 + q^2 + 2pq\cos\theta > p_F^2, \\
&\Leftrightarrow \frac{-q^2}{2pq} > \cos\theta > \frac{p_F^2 - p^2 - q^2}{2pq}.
\end{aligned} \tag{27}$$

The lower bound for $\cos\phi$ is not always larger than -1. By considering the inequality

$$\frac{p_F^2 - p^2 - q^2}{2pq} > -1 \tag{28}$$

$$\implies p_F^2 > (p - q)^2, \tag{29}$$

we find several regimes for q, with different integration limits for the ϕ-integral:

$$a) \quad 0 < q < p - p_F : \qquad \frac{-q^2}{2pq} > \cos\phi > -1, \tag{30}$$

$$b) \quad p - p_F < q < p + p_F : \qquad \frac{-q^2}{2pq} > \cos\phi > \frac{p_F^2 - p^2 - q^2}{2pq}, \tag{31}$$

$$c) \quad p + p_F < q < 2p : \qquad \frac{-q^2}{2pq} > \cos\phi > -1. \tag{32}$$

$$d) \quad 2p < q : \qquad \text{integrand always vanishes} \tag{33}$$

Combining these conditions with the condition that the argument of the delta function in functions has to be inside the integration range gives a number of different integration regions on q–θ plane. For example, integral (23) is compatible with b) and c)

Now, we do not want to calculate the three integrals (23–25) in all of these regimes. We only want to evaluate the leading one when $p - p_F \ll p_F$. For p just above the Fermi surface, the q-integration regimes a) and c) are only over a small range of q. The ranges of ϕ-integration are of similar magnitude. We may argue that b) is the most relevant region.

The condition $q > 2p_F$ for integral (23) is very restrictive for q integration and this integral is irrelevant. The same applies for (24) in which the k_x integration sets

$$-\frac{q+p_F}{p} < \cos\phi < -\frac{p_F}{p} \tag{34}$$

which restricts the angle integral close to $\cos\phi \approx -1$. The last integral (25) does not set very restrictive conditions for either q of ϕ and is the most relevant.

The integral for this region is

$$\mathrm{Im}\,\Sigma(\mathbf{p},\epsilon_p) = -\frac{g^2 m}{8\pi^3}\int_{p_F-\sqrt{2p_F^2-p^2}}^{p_F+u\sqrt{2p_F^2-p^2}} q\mathrm{d}q \tag{35}$$

$$\times\frac{1}{q}\int_{\frac{-q^2}{2pq}>\cos\phi>\frac{p_F^2-p^2-q^2}{2pq}} \mathrm{d}\phi\left[\sqrt{p_F^2-(p\cos\phi+q)^2}-\sqrt{p_F^2-(p\cos\phi)^2}\right]. \tag{36}$$

Since we are interested in the contribution to the self-energy of the lowest order in energy $\epsilon_p \propto p - p_F$, we approximate the integral

$$\begin{aligned}
&\int_{\frac{-q^2}{2pq}>\cos\phi>\frac{p_F^2-p^2-q^2}{2pq}} \mathrm{d}\phi\left[\sqrt{p_F^2-(p\cos\phi+q)^2}-\sqrt{p_F^2-(p\cos\phi)^2}\right]\\
&\approx 2\left(\arccos\frac{-2m\epsilon_p-q^2}{2pq}-\arccos\frac{-q^2}{2pq}\right)\\
&\times\left[\sqrt{p_F^2-(p\cos\phi+q)^2}-\sqrt{p_F^2-(p\cos\phi)^2}\right]\Bigg|_{\cos\phi=\frac{-2m\epsilon_p-q^2}{2pq}}\\
&\approx (2m)^2\frac{2\epsilon_p^2}{q\sqrt{4p^2-q^2}\sqrt{4p_F^2-q^2}} \approx (2m)^2\frac{2\epsilon_p^2}{q(4p_F^2-q^2)}.
\end{aligned} \tag{37}$$

The q integral can now be performed analytically, and we get

$$\begin{aligned}
\Gamma &= -\mathrm{Im}\,\Sigma(\mathbf{p},\epsilon_p)\\
&\approx \frac{3g^2m^3}{8\pi^3p_F^2}\epsilon_p^2\ln\frac{p_F}{p-p_F} \propto \epsilon_p^2\ln\frac{\mu}{\epsilon_p}.
\end{aligned} \tag{38}$$

3. Perturbation expansion for Fermi liquid interaction (15 points)

In Fermi liquid theory, the total energy of an excited state relative to the ground state energy is given by

$$E - E_0 = \sum_{\mathbf{p}\sigma}\varepsilon_{\mathbf{p}\sigma}\delta n_{\mathbf{p}\sigma} + \sum_{\mathbf{p}\sigma,\mathbf{p}'\sigma}f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\delta n_{\mathbf{p}\sigma}\delta n_{\mathbf{p}'\sigma'}, \tag{39}$$

where $\varepsilon_{\mathbf{p}\sigma}$ is the (renormalized) quasiparticle energy, $\delta n_{\mathbf{p}\sigma} = n_{\mathbf{p}\sigma} - n^{(0)}_{\mathbf{p}\sigma}$ is the difference between the occupation number $n_{\mathbf{p}\sigma}$ in an excited state and the ground state occupation number $n^{(0)}_{\mathbf{p}\sigma}$, and $f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}$ is the effective Fermi liquid interaction. The interaction can be divided into spin-dependent and spin-independent parts as

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = f^s_{\mathbf{p},\mathbf{p}'} + f^a_{\mathbf{p},\mathbf{p}'}\sigma\sigma'. \tag{40}$$

Consider the Hamiltonian

$$\hat{H} = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma}\hat{n}_{\mathbf{p}\sigma} + \frac{1}{2}\sum_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} V(|\mathbf{q}|)c^\dagger_{\mathbf{p}-\mathbf{q},\sigma}c^\dagger_{\mathbf{p}'+\mathbf{q},\sigma'}c_{\mathbf{p}'\sigma'}c_{\mathbf{p}\sigma}. \tag{41}$$

To the first order in the perturbation theory, calculate the energy of the state

$$|\Psi\rangle = |n_{\mathbf{p}_1\sigma_1}, n_{\mathbf{p}_2\sigma_2}, \ldots\rangle \tag{42}$$

which is some excited state with $n_{\mathbf{p}\sigma}$ electrons in the single-particle states $(\mathbf{p},\sigma)$. Compare the energy of this state to the noninteracting ground state. Identify the microscopic equivalents of the Landau interaction parameters f^s and f^a to the first order in the interaction. (*Hint: How to calculate the energy in the first order perturbation theory, given the interaction $\hat{V}$?*)

Solution:

Let us denote the non-interacting single-particle energy in the Hamiltonian by $\varepsilon^{(0)}_{\mathbf{p}\sigma}$ instead of $\varepsilon_{\mathbf{p}\sigma}$ to distinguish it from the renormalized one.

Calculate first the difference between the total single-particle energy between the excited state $|\Psi\rangle$ and the ground state $|g\rangle$.

$$\langle\Psi|H_0|\Psi\rangle - \langle g|H_0|g\rangle = \sum_{\mathbf{p}\sigma}\varepsilon^{(0)}_{\mathbf{p}\sigma}(n_{\mathbf{p}\sigma} - n^{(0)}_{\mathbf{p}\sigma}) = \sum_{\mathbf{p}\sigma}\varepsilon^{(0)}_{\mathbf{p}\sigma}\delta n_{\mathbf{p}\sigma} \tag{43}$$

$$\langle\Psi|\hat{V}|\Psi\rangle = \frac{1}{2}\sum_{\mathbf{p}\sigma\mathbf{p}'\sigma'\mathbf{q}} V(q)\langle\Psi|c^\dagger_{\mathbf{p}-\mathbf{q},\sigma}c^\dagger_{\mathbf{p}'+\mathbf{q},\sigma'}c_{\mathbf{p}'\sigma'}c_{\mathbf{p}\sigma}|\Psi\rangle \tag{44}$$

$$= \frac{1}{2}\sum_{\mathbf{p}\sigma\mathbf{p}'\sigma'\mathbf{q}} V(q)\langle\Psi;\mathbf{p}-\mathbf{q},\sigma;\mathbf{p}'+\mathbf{q},\sigma'|\Psi;\mathbf{p},\sigma;\mathbf{p}',\sigma'\rangle n_{\mathbf{p}'\sigma'}n_{\mathbf{p}\sigma} \tag{45}$$

$$= \frac{1}{2}\sum_{\mathbf{p}\sigma\mathbf{p}'\sigma'\mathbf{q}} V(q)\left(\delta_{\mathbf{q}=0} - \delta_{\mathbf{p}-\mathbf{q},\mathbf{p}'}\delta_{\sigma\sigma'}\right) n_{\mathbf{p}'\sigma'}n_{\mathbf{p}\sigma} \tag{46}$$

$$= \frac{1}{2}\sum_{\mathbf{p}\sigma\mathbf{p}'\sigma'} (V(0) - V(|\mathbf{p}-\mathbf{q}|)\delta_{\sigma\sigma'})\left(n^{(0)}_{\mathbf{p}'\sigma'} + \delta n_{\mathbf{p}'\sigma'}\right)\left(n^{(0)}_{\mathbf{p}\sigma} + \delta n_{\mathbf{p}'\sigma'}\right) \tag{47}$$

The terms that are linear in δn contribute to the quasiparticle energy renormalization. The quadratic terms give the interaction between excitations. We identify

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = V(0) - V(|\mathbf{p}-\mathbf{q}|)\delta_{\sigma\sigma'} \tag{48}$$

Spin symmetric and antisymmetric parts are given by

$$f^s_{\mathbf{p},\mathbf{p}'} = \frac{1}{4}\sum_{\sigma\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = V(0) - \frac{1}{2}V(|\mathbf{p}-\mathbf{q}|) \tag{49}$$

$$f^a_{\mathbf{p},\mathbf{p}'} = \frac{1}{4}\sum_{\sigma\sigma'} \sigma\sigma' f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = -\frac{1}{2}V(|\mathbf{p}-\mathbf{q}|)\delta_{\sigma\sigma'} \tag{50}$$

The first term is divergent for Coulomb interaction. This implies that Fermi liquid theory in it simplest form is only valid for short range interactions, e.g. in electrically neutral systems such as ^{3}He.