Condensed Matter Theory II: Many-Body Theory (TKM II) SoSe 2023

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Homework assignment 6
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1. Electron-phonon interaction: ($5 + 7 + 8 + 5 = 25$ points)

The Hamiltonian describing the interaction of electrons with longitudinal phonons in 3D is given by

$$\widehat{H}_{\text{e-ph}} = g \int d^3r \, \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\Phi}(\mathbf{r}),$$

where $\hat{\Phi}(\mathbf{r})$ denotes the phonon field operator. The Green's function of this phonon field is given by

$$D_0(\mathbf{q}, \omega) = \frac{\omega_{\mathbf{q}}^2}{\omega^2 - \omega_{\mathbf{q}}^2 + i0},$$

where $\omega_{\mathbf{q}} = s|\mathbf{q}|$ is the spectrum for acoustic phonons with the sound velocity s. This expression holds for $|\mathbf{q}| < k_D$, where k_D is the Debye wave-vector which determines cutoff at the Debye frequency $\omega_D = sk_D$.

(a) Consider the electron self-energy $\Sigma(\mathbf{p}, \varepsilon)$ resulting from the interaction of electrons with phonons in leading order in g and draw the self-energy diagrams. Justify why one of these diagrams does not contribute to Σ. Write down an expression for $\Sigma(\mathbf{p}, \varepsilon)$ using the Feynman rules and perform the energy integration in this expression.

(b) Consider $\operatorname{Im}\Sigma(\mathbf{p}, \varepsilon)$ for $\varepsilon \to 0$ and $p \approx p_F$. Start with an exact expression for the imaginary part of Σ. Assuming that $s \ll v_F$, justify the extension of limits for the integral over $\xi = \varepsilon_{\mathbf{p}-\mathbf{q}}$ to $\pm\infty$. Find the energy scaling of the quasiparticle decay rate at small ε in this approximation.

(c) Write down an expression for $\operatorname{Re}\Sigma(\mathbf{p}, \varepsilon)$ as an integral over the transferred momentum $\mathbf{q}$. Calculate $\operatorname{Re}\Sigma$ in the two limiting cases $\varepsilon \ll \omega_D$ and $\varepsilon \gg \omega_D$ for $p = p_F$.

(d) Use $\operatorname{Re}\Sigma(\mathbf{p}, \varepsilon)$ to determine the quasi-particle residue Z and effective mass m^*.

2. Compressibility of a two-dimensional electronic system ($5 + 5 + 10 + 5 = 25$ points)

Compressibility describes the relative change of volume of a system in response to a change in pressure

$$K(T, P, N) = -\frac{1}{V(T, P, N)} \frac{\partial V(T, P, N)}{\partial P}. \tag{1}$$

(a) Using the internal energy $E = N\varepsilon$ (ε is the average energy per particle), and $V\rho = N$ show that

$$K^{-1} = \rho^2 \frac{\partial^2}{\partial \rho^2} \rho\varepsilon(\rho) \tag{2}$$

at zero temperature.

(b) Calculate the compressibility for a 2D non-interacting electron system at zero temperature.

(c) Consider a two-dimensional electronic system at zero temperature in the presence of Coulomb interaction $U(r) = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$, where ϵ_0 is the vacuum permittivity. Calculate the interaction-induced correction to the ground state energy from the Fock diagram.

(d) Taking into account the energy correction from the Fock diagram, calculate the compressibility of the system. You should obtain

$$K^{-1} \propto \left[1 - \frac{\sqrt{2}}{\pi} r_s\right] \tag{3}$$

where $r_s = 1/\sqrt{\pi\rho a_0^2}$ is the dimensionless radius of the average volume containing one unit of charge, and $a_0 = 4\pi\epsilon_0\hbar^2/e^2 m$ is the Bohr radius and m is the electron mass.

3. Bonus: Spin susceptibility of a non-interacting electron system ($10 + 10 = 20$ bonus points)

Consider a system of non-interacting electrons. The magnetic field $\mathbf{B}(\mathbf{r}, t)$ couples to the electron spin via the term

$$H' = -g\mu_B \int \mathrm{d}\mathbf{r}\, \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{S}(\mathbf{r}), \tag{4}$$

where g is the electron g-factor, μ_B is the Bohr magneton, and the spin operator is $\mathbf{S}(\mathbf{r}) = \frac{\hbar}{2}\Psi^\dagger(\mathbf{r})\boldsymbol{\sigma}\Psi(\mathbf{r})$. Here the field operators for different spins are collected into a single vector operator

$$\Psi(\mathbf{r})^\dagger = (\psi_\uparrow^\dagger(\mathbf{r}), \psi_\downarrow^\dagger(\mathbf{r})). \tag{5}$$

and $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ is a vector of Pauli matrices.

(a) Find a formal expression for the induced spin density $\langle \mathbf{S}(\mathbf{r}, t)\rangle$ as a linear response to the applied magnetic field $\mathbf{B}$. What is the diagram associated with the expression? Express your result as a momentum integral in energy representation.

(b) Assume that the system has Coulomb interaction. Calculate the interacting spin susceptibility by taking into account the RPA diagrams as in the lectures (Sec. 3.14). What do you find?