Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 7
Deadline: 16 June 2023

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1. Plasmon dispersion in the hydrodynamic approximation ($5 + 13 + 2 = 20$ points)

Let us calculate the plasmon dispersion for a Fermi gas in the hydrodynamic limit. Use the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

for the mass density $\rho(\mathbf{r}, t) = mn(\mathbf{r}, t)$, where $\mathbf{u}(\mathbf{r}, t)$ is the velocity distribution, m is the electron mass and n is the electron number density. The force acting on the electron distribution is given by the Euler equation

$$\partial_t(\rho \mathbf{u}) = -en\mathbf{E} - \boldsymbol{\nabla} P. \tag{2}$$

where $\mathbf{E}$ is the electric field and P is the internal pressure of the Fermi gas.

(a) Due to Pauli repulsion, the pressure of a Fermi gas does not vanish at zero temperature. Use a suitable thermodynamic relation and calculate the pressure of a 3D Fermi gas with a parabolic dispersion as a function of the electron density n at zero temperature.

(b) Assume an external electric field $\mathbf{E}_{\text{ext}}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\omega t - \mathbf{q}\cdot\mathbf{r})}$ acting on the system. Calculate the linear response of the electron density $\delta n(\mathbf{q}, \omega)$ to the external field. At low frequency and long-wavelength limit, the induced electric field can be calculated from the electric scalar potential ϕ_{ind} which is determined by the Poisson equation

$$\nabla^2 \phi_{\text{ind}} = 4\pi e \delta n, \tag{3}$$

where $e > 0$. The total electric field is $\mathbf{E} = \mathbf{E}_{\text{ind}} + \mathbf{E}_{\text{ext}}$, where $\mathbf{E}_{\text{ind}}$ is the induced field given by Eq. (3). At long-wavelength limit the pressure can be calculated by locally using the relation derived in subtask (a).

(c) Determine the plasmon dispersion $\omega = \omega_{\text{p}}(\mathbf{q})$.

2. Plasmon dispersion relation from RPA (10 points)

In the lectures, the following expression was derived for the polarization bubble $\Pi(q, \omega)$ at zero temperature in three dimensions:

$$\Pi(q, \omega) = \nu \left[1 - \frac{s}{2} \ln \frac{s+1}{s-1} \right] \tag{4}$$

$$s = \frac{\omega + \mathrm{i}0\,\mathrm{sign}(\omega)}{q v_{\text{F}}} \tag{5}$$

where ν is the density of states at the Fermi surface and v_{F} is the Fermi velocity.
In the lectures, the plasmon frequency $\omega_{\text{p}}(q = 0)$ was calculated. Determine the plasmonic dispersion relation $\omega_{\text{p}}(q)$ in the limit of small q.

3. Matsubara Sums (4 + 4 + 4 + 4 + 4 = 20 points)

(a) Find the poles and residues of the Fermi and Bose distribution functions

$$n_{\mathrm{F}}(z) = \frac{1}{\exp(z\beta) + 1} \tag{6}$$

$$n_{\mathrm{B}}(z) = \frac{1}{\exp(z\beta) - 1} \tag{7}$$

assuming complex arguments $z \in \mathbb{C}$.

(b) Consider an integral of the form

$$I := \oint_{\mathcal{C}} \mathrm{d}z\, n_{\mathrm{B/F}}(z) h(z) \tag{8}$$

where $n_{\mathrm{B/F}}$ is the Bose / Fermi function, and $\oint_{\mathcal{C}} \mathrm{d}z$ an integral over a complex contour $\mathcal{C}$ which encloses all poles of $n_{\mathrm{B/F}}$ but no poles of $h(z)$.

Use the residue theorem to express I as a sum. Use your result to express a generic *Matsubara sum*

$$S := \frac{1}{\beta} \sum_{\omega_n} h(\mathrm{i}\omega_n) \tag{9}$$

in terms of a complex contour integral.

(c) Calculate the Matsubara sum

$$S(\tau) := \frac{1}{\beta} \sum_{\omega_n} g(\mathrm{i}\omega_n) \exp(\mathrm{i}\omega_n \tau) \qquad 0 \leq \tau < \beta \tag{10}$$

where $g(z)$ is holomorphic everywhere in $\mathbb{C}$ but on a countable number of points z_j. Further it holds $g(z) \lim_{|z| \to \infty} = 0$.

Calculate the sum $S(\tau)$ for both bosonic and for fermionic Matsubara frequencies by choosing an appropriate contour $\mathcal{C}$. You can assume that $g(z)$ is of the form

$$g(z) = \prod_j \frac{1}{z - z_j}. \tag{11}$$

(d) Calculate the Matsubara sums

$$S_1 = \frac{1}{\beta} \sum_{\omega_n} G_0(\mathbf{k}, \mathrm{i}\omega_n) \exp(\mathrm{i}\omega_n \tau) \tag{12}$$

$$S_2 = \frac{1}{\beta} \sum_{\omega_n} G_0(\mathbf{k}, \mathrm{i}\omega_n) G_0(\mathbf{k} + \mathbf{q}, \mathrm{i}\omega_n + \mathrm{i}\omega_m) \tag{13}$$

where

$$G_0(\mathbf{k}, \mathrm{i}\omega_m) = \frac{1}{\mathrm{i}\omega_n - \xi_{\mathbf{k}}}, \qquad \xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu, \tag{14}$$

$$\omega_n = \frac{(2n+1)\pi}{\beta}, \qquad \nu_m = \frac{2m\pi}{\beta}. \tag{15}$$

(e) Consider sum (10) again. Assume, that $g(z)$ is analytic everywhere, but on the real axis. Choose an appropriate contour to express $S(\tau)$ as an integral

$$S(\tau) = \epsilon \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} n_{\mathrm{B/F}}(\omega) a(\omega) \exp(\omega\tau) \qquad 0 < \tau < \beta \tag{16}$$

where $\epsilon = -1$ for bosons, $\epsilon = 1$ for fermions, $a(\omega) = \mathrm{i}(g(\omega + \mathrm{i}\delta) - g(\omega - \mathrm{i}\delta))$.