Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 9
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1. Fluctuation-dissipation theorem (5 + 10 = 15 points)

In linear response theory, fluctuation-dissipation theorem relates the fluctuations of an observable to the dissipative part of a response function. An early form was first derived by Einstein (1905) in the context of Brownian motion.

Let us derive the theorem using imaginary-time formalism. Let $\hat{A}$ be some observable, and the interaction picture operator is defined as $\hat{A}(t) = e^{iH_0 t/\hbar}\hat{A}e^{-iH_0 t/\hbar}$ for any $t \in \mathbb{C}$.

(a) Consider a noise correlation function $S(t) = \langle \hat{A}(t)\hat{A}(0)\rangle$, where the expectation value denotes the thermal trace $\langle \cdots \rangle = \mathrm{Tr}\left[e^{\beta H_0} \cdots\right]$. Show that the noise correlation function satisfies the Kubo-Martin-Schwinger (KMS) relation

$$S(t) = S(-t - i\hbar\beta). \tag{1}$$

(b) Fourier transform the KMS relation and find the relation between $S(\omega)$ and $S(-\omega)$.

(c) Consider an external perturbation f which couples to $\hat{A}$. The response function which describes the effect of f on $\langle A\rangle$ is given by the Kubo formula (Eq. (4.107) in the lectures)

$$\mathcal{D}^R_{AA}(t) = -i\Theta(t)\langle[\hat{A}(t), \hat{A}(0)]\rangle. \tag{2}$$

Using Eqs.(1) and (2), prove the fluctuation-dissipation theorem

$$S(\omega) = 2\hbar[1 + \coth(\beta\hbar\omega/2)]\,\mathrm{Im}\,[\mathcal{D}^R_{AA}(\omega)]. \tag{3}$$

Solution:

(a) The Kubo-Martin-Schwinger condition is obtained by using the Heisenberg representation and the cyclic property of the trace:

$$\begin{aligned} S(t) &= \mathrm{Tr}\left[\overbrace{e^{-\beta H_0}e^{iH_0 t/\hbar}}^{e^{iH_0(t+i\hbar\beta)/\hbar}}\hat{A}(0)e^{-iH_0 t/\hbar}\hat{A}(0)\right] \\ &= \mathrm{Tr}\left[e^{iH_0(t+i\hbar\beta)/\hbar}\hat{A}(0)e^{-\beta H_0}e^{-iH_0(t+i\hbar\beta)/\hbar}\hat{A}(0)\right]. \\ &= \mathrm{Tr}\left[e^{-\beta H_0}e^{-iH_0(t+i\hbar\beta)/\hbar}\hat{A}(0)e^{iH_0(t+i\hbar\beta)/\hbar}\hat{A}(0)\right] \\ &= \langle A(-t-i\hbar\beta)A(0)\rangle = S(-t-i\hbar\beta) \end{aligned} \tag{4}$$

(b) Let us make a Fourier transform and shift the integration contour. We assume that $S(t)$ vanishes at infinity so we do not need to include the vertical parts connecting the two integration lines.

$$
\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} S(t) \\
&= \int_{-\infty}^{\infty} dt e^{i\omega t} S(-t - i\hbar\beta) \\
&= \int_{-\infty}^{\infty} ds e^{-i\omega s} e^{\beta\hbar\omega} S(s) \\
&= e^{\beta\hbar\omega} S(-\omega).
\end{aligned} \tag{5}
$$

This is the detailed balance condition.

(c) Let us define C such that

$$C(t) \equiv \frac{1}{2i}\langle[\hat{A}(t), \hat{A}(0)]\rangle = -\frac{1}{2i}\langle[\hat{A}(-t), \hat{A}(0)]\rangle = -C(-t) \tag{6}$$

$$C(t) \equiv \frac{1}{2i}\langle[\hat{A}(t), \hat{A}(0)]\rangle = \frac{\langle\hat{A}(t)\hat{A}(0)\rangle - \langle\hat{A}(-t)\hat{A}(0)\rangle}{2i} = \frac{S(t) - S(-t)}{2i} \tag{7}$$

$$C(\omega) = \frac{S(\omega) - S(-\omega)}{2i} = \frac{1 - e^{\beta\hbar\omega}}{2i} S(\omega) = \frac{-iS(\omega)}{\coth(\beta\hbar\omega/2) + 1} \tag{8}$$

According to Kubo formula, the response function is $\mathcal{D}^R_{AA}(t) = -2\Theta(t)C(t)/\hbar$.

$$
\begin{aligned}
C(\omega) = \int \mathrm{d}t e^{i\omega t} C(t) &= \int_{-\infty}^{0} \mathrm{d}t e^{i\omega t} C(t) + \int_{0}^{\infty} \mathrm{d}t e^{i\omega t} C(t) \\
&= \int \mathrm{d}t e^{i\omega t} \Theta(t) C(t) + \int \mathrm{d}t e^{-i\omega t} \Theta(t) \underbrace{C(-t)}_{-C(t)} \\
&= -\frac{\hbar}{2}\left(\mathcal{D}^R_{AA}(\omega) - \mathcal{D}^R_{AA}(-\omega)\right) = -i\hbar \mathrm{Im}\, \mathcal{D}^R_{AA}(\omega)
\end{aligned} \tag{9}
$$

Combining the above two results, we obtain

$$S(\omega) = 2\hbar[1 + \coth(\beta\hbar\omega/2)]\, \mathrm{Im}\,[\mathcal{D}^R_{AA}(\omega)]. \tag{10}$$

Wick's theorem from Gaussian integrals: (5 + 5 = 10 points)

(a) In the lectures, Gaussian integrals over complex fields $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{C}^N$ with the integration measure $d(\mathbf{x}^\dagger, \mathbf{x}) = \prod_{i=1}^{N}(d\mathrm{Re}\, x_i\, d\mathrm{Im}\, x_i)$ were discussed. Let A be an $N \times N$ complex positive-definite matrix. The average is defined as

$$\langle \ldots \rangle = \frac{\int d(\mathbf{x}^\dagger, \mathbf{x})\, (\ldots)\, e^{-\mathbf{x}^\dagger A \mathbf{x}}}{\int d(\mathbf{x}^\dagger, \mathbf{x})\; e^{-\mathbf{x}^\dagger A \mathbf{x}}}\,, \tag{1}$$

Prove the Wick's theorem for complex ("bosonic") fields:

$$\langle x^*_{j_1} x^*_{j_2} \ldots x^*_{j_n} x_{k_1} x_{k_2} \ldots x_{k_n} \rangle = \sum_{\text{permutations } P} \langle x^*_{j_1} x_{k_{p_1}} \rangle \langle x^*_{j_2} x_{k_{p_2}} \rangle \ldots \langle x^*_{j_n} x_{k_{p_n}} \rangle\,, \tag{2}$$

where the sum goes over permutations $P = \{p_1, \ldots, p_n\}$ of $\{1, \ldots, n\}$.

Solution:

For simplicity, we assume that the matrix A is Hermitian; the proof can be extended onto non-Hermitian matrices with a positive-definite Hermitian part $(A + A^\dagger)/2$. The starting point is the definition of the function of the source vectors $\mathbf{J}_1$ and $\mathbf{J}_2$,

$$Z[\mathbf{J}] := Z[\mathbf{J}_1, \mathbf{J}_2] = \int d(\mathbf{x}^\dagger, \mathbf{x})\; e^{-\mathbf{x}^\dagger A \mathbf{x} + \mathbf{J}_1^\dagger \mathbf{x} + \mathbf{x}^\dagger \mathbf{J}_2}. \tag{3}$$

We notice that the denominator of the expectation value is given by $Z[\mathbf{J} = 0]$ and differentiating with respect to the source variables (elements of vectors $\mathbf{J}_1^\dagger$ and $\mathbf{J}_2$) leads to

$$\frac{\partial Z[\mathbf{J}]}{\partial J_{2,i}} = \int d(\mathbf{x}^\dagger, \mathbf{x})\, x_i^*\, e^{-\mathbf{x}^\dagger A \mathbf{x} + \mathbf{J}_1^\dagger \mathbf{x} + \mathbf{x}^\dagger \mathbf{J}_2}, \tag{4}$$

$$\frac{\partial Z[\mathbf{J}]}{\partial J^*_{1,i}} = \int d(\mathbf{x}^\dagger, \mathbf{x})\, x_i\, e^{-\mathbf{x}^\dagger A \mathbf{x} + \mathbf{J}_1^\dagger \mathbf{x} + \mathbf{x}^\dagger \mathbf{J}_2}. \tag{5}$$

Using these formulas for general averages, we get

$$\langle x^*_{j_1} x^*_{j_2} \ldots x^*_{j_n} x_{k_1} x_{k_2} \ldots x_{k_n} \rangle = \frac{1}{Z[\mathbf{J} = 0]} \times \left(\frac{\partial}{\partial J^*_{1,k_n}} \cdots \frac{\partial}{\partial J^*_{1,k_2}} \frac{\partial}{\partial J^*_{1,k_1}} \frac{\partial}{\partial J_{2,j_n}} \cdots \frac{\partial}{\partial J_{2,j_2}} \frac{\partial}{\partial J_{2,j_1}} Z[\mathbf{J}] \right)_{\mathbf{J}=0}. \tag{6}$$

The generating function (partition function in the presence of the sources) $Z[\mathbf{J}]$ can be evaluated analytically, see Eq. (6.50) of the Lecture Notes and Appendix below:

$$Z[\mathbf{J}] = \frac{\pi^N}{\det(A)} e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} = \frac{\pi^N}{\det(A)} \exp\left[\sum_{l,m} J^*_{1,l} \left(A^{-1}\right)_{lm} J_{2,m} \right]. \tag{7}$$

From this we see that

$$\frac{\partial}{\partial J_{1,k}^*}\frac{\partial}{\partial J_{2,j}}Z[\mathbf{J}] = \frac{\partial}{\partial J_{1,k}^*}\left\{\frac{\partial}{\partial J_{2,j}}Z[\mathbf{J}]\right\} = \frac{\partial}{\partial J_{1,k}^*}\left\{\left[\sum_p J_{1,l}^*\left(A^{-1}\right)_{lj}\right]Z[\mathbf{J}]\right\}$$
$$= \left\{Z[\mathbf{J}]\frac{\partial}{\partial J_{1,k}^*}\left[\sum_l J_{1,l}^*\left(A^{-1}\right)_{lj}\right] + \left[\sum_l J_{1,l}^*\left(A^{-1}\right)_{lj}\right]\frac{\partial Z[\mathbf{J}]}{\partial J_{1,k}^*}\right\}$$
$$= Z[\mathbf{J}]\left\{\left(A^{-1}\right)_{kj} + \left[\sum_l J_{1,l}^*\left(A^{-1}\right)_{lj}\right]\left[\sum_m \left(A^{-1}\right)_{km}J_{2,m}\right]\right\}. \qquad (8)$$

Setting $\mathbf{J}=0$, we remove the second term in the curly brackets – the one coming from $\partial Z[\mathbf{J}]/\partial J_{1,k}^*$, and thus obtain Eq. (6.52) of the Lecture Notes:

$$\langle x_j^* x_k\rangle = \left(A^{-1}\right)_{kj}. \qquad (9)$$

Now we consider the derivatives in the higher-order correlators in Eq. (6):

$$\frac{\partial}{\partial J_{1,k_n}^*}\cdots\frac{\partial}{\partial J_{1,k_2}^*}\frac{\partial}{\partial J_{1,k_1}^*}\frac{\partial}{\partial J_{2,j_n}}\cdots\frac{\partial}{\partial J_{2,j_2}}\frac{\partial}{\partial J_{2,j_1}}Z[\mathbf{J}]\Bigg|_{\mathbf{J}=0}$$
$$= \frac{\partial}{\partial J_{1,k_n}^*}\cdots\frac{\partial}{\partial J_{1,k_2}^*}\frac{\partial}{\partial J_{1,k_1}^*}\left\{\left[\sum_{l_1} J_{1,l_1}^*\left(A^{-1}\right)_{l_1 j_1}\right]\cdots\left[\sum_{l_n} J_{1,l_n}^*\left(A^{-1}\right)_{l_n j_n}\right]Z[\mathbf{J}]\right\}\Bigg|_{\mathbf{J}=0}. \qquad (10)$$

As in the two-point correlator, Eq. (8), each derivative may act on the product of square brackets or on $Z[\mathbf{J}]$, but the latter contribution vanishes at $\mathbf{J}=0$. Thus, we are left with the product of the former terms, each giving a matrix element $(A^{-1})_{k,j}$. Since for a given k_i we can find J_{1,k_i}^* in any of the square brackets in the product, we generate all possible permutations of the second index in the matrix A^{-1} and should sum over all these permutations:

$$\langle x_{j_1}^* x_{j_2}^* \ldots x_{j_n}^* x_{k_1} x_{k_2} \ldots x_{k_n}\rangle = \frac{1}{Z[\mathbf{J}=0]}$$
$$\times\frac{\pi^N}{\det(A)}\sum_{\text{permutations } P}\left(A^{-1}\right)_{k_1,j_{P_1}}\left(A^{-1}\right)_{k_2,j_{P_2}}\cdots\left(A^{-1}\right)_{k_n,j_{P_n}}, \qquad (11)$$

which, combined with Eq. (9), proves Wick's theorem.

(b) For Grassmann variables, the average is defined as

$$\langle\ldots\rangle = \frac{\int d(\eta^*,\eta)\,(\ldots)\,e^{-\eta^{*T}A\,\eta}}{\int d(\eta^*,\eta)\;e^{-\eta^{*T}A\,\eta}}. \qquad (12)$$

Prove the Wick's theorem for Grassmann ("fermionic") fields:

$$\langle\eta_{j_1}\eta_{j_2}\ldots\eta_{j_n}\eta_{k_n}^*\ldots\eta_{k_2}^*\eta_{k_1}^*\rangle \;=\; \sum_{\text{permutations } P}\text{sgn}(P)\;\langle\eta_{j_1}\eta_{k_{p_1}}^*\rangle\langle\eta_{j_2}\eta_{k_{p_2}}^*\rangle\ldots\langle\eta_{j_n}\eta_{k_{p_n}}^*\rangle, \qquad (13)$$

where the sum goes over permutations $P=\{p_1,\ldots,p_n\}$ of $\{1,\ldots,n\}$, and $\text{sgn}(P)$ is the sign of the permutation.

Solution:
We start with Eq. (6.79) of the Lecture Notes (see Appendix for the details of derivation) and continue with

$$\int d(\eta^*,\eta)\exp\left(-\eta^{*T}A\eta+\xi^{*T}\eta+\eta^{*T}\xi\right)=\det(A)e^{\xi^{*T}A^{-1}\xi}. \tag{14}$$

This identity can be shown easily by remembering that the integral is shift-invariant under $\eta\to\tilde\eta=\eta-A^{-1}\xi$, $\eta^*\to\tilde\eta^*=\eta^*-\xi^*A^{-T}$:

$$\begin{aligned}&\int d(\eta^*,\eta)\exp\left(-\eta^{*T}A\eta+\xi^{*T}\eta+\eta^{*T}\xi-\xi^{*T}A\xi\right)\\&=\int d(\eta^*,\eta)\exp\left(-(\eta^{*T}-\xi^{*T}A^{-1})A(\eta-A^{-1}\xi)\right)\\&=\int d(\tilde\eta^*,\tilde\eta)\exp\left(-\tilde\eta^{*T}A\tilde\eta\right)=\det(A)\end{aligned} \tag{15}$$

We can write generic n point functions as n-th derivatives of the partition function

$$\begin{aligned}Z[\xi^*,\xi]&\equiv Z[0,0]^{-1}\int d(\eta^*,\eta)\exp\left(-\eta^{*T}A\eta+\xi^{*T}\eta+\eta^{*T}\xi\right),\\ \langle\eta^*_{i_1}\eta_{j_1}\dots\eta^*_{i_n}\eta_{j_n}\rangle&=\partial_{\xi^*_{i_1}}\partial_{\xi_{j_1}}\dots\partial_{\xi^*_{i_n}}\partial_{\xi_{j_n}}Z[\xi^*,\xi].\end{aligned} \tag{16}$$

We then use $Z[\xi^*,\xi]=e^{\xi^{*T}A^{-1}\xi}$ to find, similarly to the derivation in subtask 1a:

$$\langle\eta^*_{i_1}\eta_{j_1}\dots\eta^*_{i_n}\eta_{j_n}\rangle=\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)A^{-1}_{i_1\sigma_{i_1}}\dots A^{-1}_{i_n\sigma_{i_n}} \tag{17}$$

With $\langle\eta^*_i\eta_j\rangle=A^{-1}_{ij}$, this demonstrates the validity of Wick's theorem.

2. **Fermionic coherent states:** (5 + 5 + 5 + 5 = 20 points)

Prove the following identities introduced in the lecture for fermionic coherent states in terms of Grassmann variables (Sec. 6.2.3 of Lecture Notes):

(a) Right action of $a_k^\dagger$ and left action of a_k:

$$a_k^\dagger|\eta\rangle=-\frac{\partial}{\partial\eta_k}|\eta\rangle\,,\qquad\langle\eta|a_k=\frac{\partial}{\partial\eta_k^*}\langle\eta|\,. \tag{18}$$

Solution:
The definition of the fermionic coherent state is:

$$|\eta\rangle=\exp\left[-\sum_k\eta_k a_k^\dagger\right]|0\rangle. \tag{19}$$

Remembering that the terms under the sum commute with each other, we can pull the sum out of the exponential. Then we can use that Grassman variables square to zero and write:

$$|\eta\rangle=\prod_k\left[1-\eta_k a_k^\dagger\right]|0\rangle. \tag{20}$$

The creators/annihilators $a_k^\dagger, a_k$ anticommute with the Grassman variables. Therefore, the product $\eta_{k'}a_{k'}^\dagger$ commutes with $\eta_k, \partial_{\eta_k}$ for $k \neq k'$:

$$\begin{aligned}
a_k^\dagger|\eta\rangle &= \prod_{k'<k}\left[1-\eta_{k'}a_{k'}^\dagger\right]\left[a_k^\dagger-\eta_k a_k^\dagger a_k^\dagger\right]\prod_{k<k'}\left[1-\eta_{k'}a_{k'}^\dagger\right]|0\rangle \\
&= \prod_{k'<k}\left[1-\eta_{k'}a_{k'}^\dagger\right](-\partial_{\eta_k})\left[1-\eta_k a_k^\dagger\right]\prod_{k<k'}\left[1-\eta_{k'}a_{k'}^\dagger\right]|0\rangle \\
&= -\frac{\partial}{\partial\eta_k}|\eta\rangle. \qquad (21)
\end{aligned}$$

The other equation follows by Hermitean conjugation and renaming to the other set of independent variables η_k^*. Note the flipped sign since the derivative switches place with a_k once.

(b) Overlap (here η_k and ψ_k^* are two sets of Grassmann variables):

$$\langle\psi|\eta\rangle = \exp\left(\sum_k \psi_k^*\eta_k\right). \qquad (22)$$

Solution:

We can re-order the product, using that pairs of Grassman variables mutually commute:

$$\begin{aligned}
\langle\psi|\eta\rangle &= \langle 0|\prod_k\left[1-a_k\psi_k^*\right]\prod_{k'}\left[1-\eta_{k'}a_{k'}^\dagger\right]|0\rangle \\
&= \langle 0|\prod_k\left[(1-a_k\psi_k^*)\left(1-\eta_k a_k^\dagger\right)\right]|0\rangle \\
&= \langle 0|\prod_k\left[\left(1-a_k\psi_k^*-\eta_k a_k^\dagger+\psi_k^*\eta_k a_k a_k^\dagger\right)\right]|0\rangle. \qquad (23)
\end{aligned}$$

Then we can use that $a_k a_k^\dagger = n_k+1$ and $n_k|0\rangle = 0$. Unpaired creators or annihilators do not create overlap with $\langle 0|$:

$$\begin{aligned}
&= \langle 0|\prod_k\left[1+\psi_k^*\eta_k\right]|0\rangle \\
&= \prod_k\left[1+\psi_k^*\eta_k\right] \\
&= \exp\left[\sum_k \psi_k^*\eta_k\right]. \qquad (24)
\end{aligned}$$

(c) Completeness:

$$\int \mathcal{D}(\eta^*,\eta)\exp\left[-\sum_k \eta_k^*\eta_k\right]|\eta\rangle\langle\eta| = 1 \qquad \text{with} \quad \mathcal{D}(\eta^*,\eta) = \prod_j d\eta_j^* d\eta_j\,. \qquad (25)$$

Solution:

We put the definitions of the coherent state into the equation:

$$\int \mathcal{D}(\eta^*,\eta)\prod_{k_1}\left[1-\eta_{k_1}^*\eta_{k_1}\right]\prod_{k_2}\left[1-\eta_{k_2}a_{k_2}^\dagger\right]|0\rangle\langle 0|\prod_{k_3}\left[1-a_{k_3}\eta_{k_3}^*\right]. \qquad (26)$$

Then we rewrite the products as sums over series $\{n_k\}_k$ with $n_k \in \{0,1\}$:

$$\begin{aligned}\prod_{k_1}\left[1-\eta^*_{k_1}\eta_{k_1}\right] &= \sum_{\{n_k\}_k}\prod_k\left(-\eta^*_k\eta_k\right)^{1-n_k}\\ \prod_{k_3}\left[1-\eta^*_{k_3}a_{k_3}\right] &= \sum_{\{n_k\}_k}\prod_k\left(-\eta^*_k a_k\right)^{n_k}\,.\end{aligned}\tag{27}$$

We therefore have to sum over three distinct series, one for each sum. In the Grassman integral only terms with all variables contribute:

$$\int\mathcal{D}(\eta^*,\eta)\sum_{\{n_k\}_k}\prod_k\left(-\eta^*_k\eta_k\right)^{1-n_k}\prod_{k_2}\left(-\eta_{k_2}a^\dagger_{k_2}\right)^{n_{k_2}}|0\rangle\langle 0|\prod_{k_3}\left(-a_{k_3}\eta^*_{k_3}\right)^{n_{k_3}}\,.\tag{28}$$

We re-order η and a. The $-\eta^*$ variables move past a and thus flip the sign; the η variables pass bilinear combinations of $a^\dagger\eta$ and do not change sign:

$$\int\mathcal{D}(\eta^*,\eta)\sum_{\{n_k\}_k}\prod_k\left(-\eta^*_k\eta_k\right)^{1-n_k}\prod_k\left(-\eta_k\eta^*_k\right)^{n_k}\prod_{k_2}\left(a^\dagger_{k_2}\right)^{n_{k_2}}|0\rangle\langle 0|\prod_{k_3}\left(a_{k_3}\right)^{n_{k_3}}\,.\tag{29}$$

This can be written as

$$\int\mathcal{D}(\eta^*,\eta)\prod_k\left(-\eta^*_k\eta_k\right)\cdot\sum_{\{n_k\}_k}|\{n_k\}_k\rangle\langle\{n_k\}_k|,\tag{30}$$

which is easily recognized as a partition of unity in particle number basis after the integral evaluates to one.

(d) Trace of an arbitrary operator $\hat{A}$ in terms of coherent-state matrix elements:

$$\mathrm{Tr}\,\hat{A} = \int\mathcal{D}(\eta^*,\eta)\,e^{-\sum_k\eta^*_k\eta_k}\langle-\eta|\hat{A}|\eta\rangle\,.\tag{31}$$

Solution:

First, we need to show auxiliary identities:

$$\mathrm{Tr}\left(|\eta\rangle\langle\eta|\hat{A}\right) = \langle-\eta|\hat{A}|\eta\rangle\tag{32}$$

$$\langle\{n_k\}_k|\eta\rangle = \prod_k(-\eta_k)^{n_k}\tag{33}$$

$$\langle\eta|\{n_k\}_k\rangle = \prod_k(-\eta^*_k)^{n_k}.\tag{34}$$

We take the trace and use the occupation number basis. We then insert unity. We can label a complete and orthogonal states by the occupation numbers:

$$\begin{aligned}\mathrm{Tr}\left(|\eta\rangle\langle\eta|\hat{A}\right) &= \sum_{\{n_k\}_k}\langle\{n_k\}_k|\eta\rangle\langle\eta|\left(\sum_{\{n_{k'}\}_{k'}}|\{n_{k'}\}_{k'}\rangle\langle\{n_{k'}\}_{k'}|\right)\hat{A}|\{n_k\}_k\rangle\\ &= \sum_{\{n_k\}_k}\sum_{\{n_{k'}\}_{k'}}\langle\{n_k\}_k|\eta\rangle\langle\eta|\{n_{k'}\}_{k'}\rangle\langle\{n_{k'}\}_{k'}|\hat{A}|\{n_k\}_k\rangle\\ &= \sum_{\{n_k\}_k}\sum_{\{n_{k'}\}_{k'}}\langle-\eta|\{n_{k'}\}_{k'}\rangle\langle\{n_{k'}\}_{k'}|\hat{A}|\{n_k\}_k\rangle\langle\{n_k\}_k|\eta\rangle.\end{aligned}\tag{35}$$

When moving the matrix elements of coherent states (34) past each other, we pick up a $(-1)^{\sum_k n_k}$. This means $(-1)^{\sum_k n_k}\langle\eta|\{n_k\}_k\rangle = \prod_k \eta^{n_k} = \langle -\eta|\{n_k\}_k\rangle$.
Now we employ the identity from above and use the linearity of the integrals:

$$
\begin{aligned}
\mathrm{Tr}\hat{A} &= \int \mathcal{D}(\eta^*, \eta) \exp\left[-\sum_k \eta_k^* \eta_k\right] \mathrm{Tr}\left(|\eta\rangle\langle\eta|\hat{A}\right) \\
&= \int \mathcal{D}(\eta^*, \eta) \exp\left[-\sum_k \eta_k^* \eta_k\right] \langle -\eta|\hat{A}|\eta\rangle.
\end{aligned} \tag{36}
$$

Appendix 1a

Here we demonstrate in detail how Eq. (7) is derived. Since the matrix A is diagonalizable, writing $A = U^\dagger D U$, we get

$$Z[\mathbf{J}] = \int d(\mathbf{x}^\dagger, \mathbf{x})\; e^{-\mathbf{x}^\dagger U^\dagger D U \mathbf{x} + \mathbf{J}_1^\dagger \mathbf{x} + \mathbf{x}^\dagger \mathbf{J}_2}. \tag{96}$$

Next, we define $\mathbf{y} = U\mathbf{x}$ and, since U is unitary, the functional determinant of this transformation has the absolute value 1. Then we get

$$Z[\mathbf{J}] = \int d(\mathbf{y}^\dagger, \mathbf{y})\; e^{-\mathbf{y}^\dagger D \mathbf{y} + \mathbf{J}_1^\dagger U^\dagger \mathbf{y} + \mathbf{y}^\dagger U \mathbf{J}_2}. \tag{97}$$

In order to get rid of the linear terms we define again new integration variables $\mathbf{y} = \mathbf{z} + D^{-1} U \mathbf{J}_2$ and $\mathbf{y}^\dagger = \mathbf{z}^\dagger + \mathbf{J}_1^\dagger U^\dagger D^{-1}$:

$$Z[\mathbf{J}] = e^{\mathbf{J}_1^\dagger U^\dagger D^{-1} U \mathbf{J}_2} \int d(\mathbf{z}^\dagger, \mathbf{z})\; e^{-\mathbf{z}^\dagger D \mathbf{z}} \tag{98}$$

Note that this does not require $\mathbf{J}_1 = \mathbf{J}_2$, because we integrate over the imaginary and the real part independently, which is equivalent to integrating over the complex vector and its conjugate independently.

Now we recall that $A = U^\dagger D U \Leftrightarrow A^{-1} = U^\dagger D^{-1} U$ and express the integral through the components of the vectors

$$\begin{aligned} Z[\mathbf{J}] &= e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} \int \prod_{i=1}^{N} (d\mathrm{Re}\, z_i\, d\mathrm{Im}\, z_i)\; e^{-\sum_{ij} z_i^* D_{ij} z_j} \underbrace{=}_{D_{ij} \propto \delta_{ij}} e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} \int \prod_{i=1}^{N} (d\mathrm{Re}\, z_i\, d\mathrm{Im}\, z_i)\; e^{-\sum_i z_i^* D_{ii} z_i} \\ &= e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} \prod_{i=1}^{N} \int (d\mathrm{Re}\, z_i\, d\mathrm{Im}\, z_i)\; e^{-D_{ii} |z_i|^2} = e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} \prod_{i=1}^{N} \int (d\mathrm{Re}\, z_i\, d\mathrm{Im}\, z_i)\; e^{-D_{ii} [\mathrm{Re}\,(z_i)^2 + \mathrm{Im}\,(z_i)^2]} \\ &= e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} \prod_{i=1}^{N} \int d\mathrm{Re}\, z_i\; e^{-D_{ii} \mathrm{Re}\,(z_i)^2} \int d\mathrm{Im}\, z_i\; e^{-D_{ii} \mathrm{Im}\,(z_i)^2} = e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} \prod_{i=1}^{N} \sqrt{\frac{\pi}{D_{ii}}} \sqrt{\frac{\pi}{D_{ii}}} \end{aligned} \tag{99}$$

Since A is positive definite it has positive eigenvalues and we get

$$Z[\mathbf{J}] = e^{\mathbf{J}_1^\dagger A^{-1} \mathbf{J}_2} \frac{\pi^N}{\det(A)}. \tag{100}$$

Appendix 1b

Here, we demonstrate how Eq. (6.79) of the Lecture Notes,

$$\int d(\eta^*, \eta) \exp\left(-\eta^{*T} A \eta\right) = \det(A), \tag{101}$$

is derived. For this purpose we expand the exponential and keep only the N-th order term. Only this order can contribute under the integral:

$$\begin{aligned}
\left(-\eta^{*T} A \eta\right)^N &= \sum_{i_1, j_1 = 1}^{N} \cdots \sum_{i_N, j_N = 1}^{N} \left(-\eta^{*T}_{i_1} A_{i_1 j_1} \eta_{j_1}\right) \ldots \left(-\eta^{*T}_{i_N} A_{i_N j_N} \eta_{j_N}\right) \\
&= \sum_{\sigma, \tau \in S_N} \left(\prod_{k=1}^{N} A_{\sigma_k \tau_k}\right) \prod_{k=1}^{N} \left(-\eta^{*T}_{\sigma_k} \eta_{\tau_k}\right) \\
&= \sum_{\sigma, \tau \in S_N} \mathrm{sign}(\tau)\mathrm{sign}(\sigma) \left(\prod_{k=1}^{N} A_{\sigma_k \tau_k}\right) \prod_{k=1}^{N} \left(-\eta^{*T}_{k} \eta_{k}\right) \\
&= \sum_{\tau \in S_N} \det(A) \prod_{k=1}^{N} \left(-\eta^{*T}_{k} \eta_{k}\right).
\end{aligned} \tag{102}$$

We are left with the sum over τ that gives a factor $N!$ which is cancelled since it is the N-th order of the expansion.