Condensed Matter Theory II: Many-Body Theory (TKM II) SoSe 2023

PD Dr. I. Gornyi and Prof. Dr. A. Mirlin
Dr. Risto Ojajärvi and Paul Pöpperl

Homework assignment 10
Deadline: 7 July 2023

---

1. Effective action for electron-phonon system $(5 + 10 + 2 + 8 = 25$ points)

Consider electrons interacting with phonons. The system is described by the Hamilton operator $\hat{H} = \hat{H}_{\rm el} + \hat{H}_{\rm ph} + \hat{H}_{\rm el-ph}$, where

$$\hat{H}_{\rm el} = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}}\, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\,, \qquad \hat{H}_{\rm ph} = \sum_{\mathbf{q}} \omega_q\, b^\dagger_{\mathbf{q}} b_{\mathbf{q}}\,, \tag{1}$$

describe free electrons and phonons, and

$$\hat{H}_{\rm el-ph} = g \sum_{\mathbf{p},\mathbf{q}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}+\mathbf{q}}\, i\sqrt{\omega_q}\,(b_{\mathbf{q}} - b^\dagger_{-\mathbf{q}}) \tag{2}$$

describes the electron-phonon interaction. In this task, we express the partition function of the system, as well as Green's functions, as functional integrals.

(a) Express the partition function of the system as a functional integral over fermionic and bosonic fields.

(b) Integrate out the bosonic field configurations in the partition function to derive an effective action for fermions

(c) Derive the free electron Green's function from the partition function for non-interacting electrons. Introduce source fields in the action and take derivatives of the partition function with respect to them.

(d) Expand the partition function to first order in g^2 and derive an expression for the lowest-order correction to the fermionic Green's function induced by the electron-phonon interaction.

2. Hubbard-Stratonovich transformation $(1 + 7 + 7 + 2 + 3 + 5 = 25$ points)

Assume that we have a general electron-electron interaction

$$H^{\rm int}(\psi^*, \psi) = \frac{1}{2} \sum_{abcd} V_{ad,bc} \psi^*_a \psi^*_b \psi_c \psi_d, \tag{3}$$

where a, b, c, d refer to *e.g.* spin σ and space-time coordinates $(\mathbf{r}, \tau)$ or $(\mathbf{p}, \omega_n)$. Within the path integral, we can introduce some suitable set of bilinear operators $\rho_n = \psi^*_a \psi_d$ and $\rho_m = \psi^*_b \psi_c$ to write the interaction as

$$H^{\rm int}(\psi^*, \psi) = \frac{1}{2} \sum_{nm} \rho_n V_{nm} \rho_m. \tag{4}$$

It is then possible to introduce a new (real) bosonic field ϕ, and express the interaction as

$$\exp\left(-\frac{1}{2}\sum_{nm}\rho_m V_{mn}\rho_n\right) = \mathcal{N}\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\sum_{nm}\phi_m V_{mn}^{-1}\phi_n - \sum_m i\phi_m\rho_m\right), \tag{5}$$

where the prefactor $\mathcal{N}$ does not contain any fields and does not affect the dynamics. This is similar to using Eq. (6.55) in reverse. However, we have introduced an extra imaginary unit to obtain the negative sign on the left-hand side. This is necessary in case of repulsive interaction ($V_{nm} > 0$). For an attractive interaction we can use Eq. (6.55) as it stands.

The equation (5) is known as the Hubbard-Stratonovich transformation, and it allows us to express any electron-electron interaction as an interaction between an electron and a Gaussian bosonic field ϕ. The transformation is exact; new action is completely equivalent to the original one, but does not contain a quartic electronic term anymore. The price we pay is that there is an extra field with its own dynamics. The power of the Hubbard-Stratonovich transformation comes from the fact that it allows us to make approximations systematically.

Let us study the fermionic action for electrons interacting through the Coulomb potential, and use the Hubbard-Stratonovich transformation to derive the effective RPA interaction between the electrons.

(a) To begin, write down the action for electrons interacting via the Coulomb potential.

(b) For the Coulomb interaction, make a Hubbard-Stratonovich transformation by introducing a bosonic field ϕ that couples to the electron density

$$\rho(\mathbf{q},\tau) = \sum_{\mathbf{p},\sigma} \psi_\sigma^*(\mathbf{p},\tau)\psi_\sigma(\mathbf{p}+\mathbf{q},\tau). \tag{6}$$

(c) After the Hubbard-Stratonovich transformation, the electronic action is quadratic. Integrate out the fermions using Eq. (6.80) and obtain an effective action $S_{\text{eff}}[\phi]$ that only includes the field ϕ as a variable.

(d) The dominant contribution to the functional integral comes from the vicinity of the minimum of the action $S_{\text{eff}}[\phi]$. Find the minimum by taking a functional derivative

$$\frac{\delta S_{\text{eff}}[\phi]}{\delta\phi} = 0. \tag{7}$$

The minimum corresponds to the mean-field value of the field ϕ.

(e) The low-energy dynamics of the field are given by the fluctuations of the field ϕ around its mean-field value. Expand the action to quadratic order in ϕ and identify the polarization operator and the screened Coulomb interaction U_{eff}^{-1}.

(f) Using the quadratic approximation for the effective action written in terms of the Hubbard-Stratonovich fields, express the RPA free energy of the interacting electron gas in terms of the polarization operator.