Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 11
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1. Ginzburg-Landau action $(10+1+4+10=25$ points$)$

Consider the superconducting Ginzburg-Landau action

$$S[\Delta,\Delta^*]=\beta\int \mathrm{d}^3 r\left[a(T)|\Delta|^2+b|\Delta|^4+K|\partial_{\mathbf{r}}\Delta|^2\right]. \tag{1}$$

which was discussed at Sec. 6.3.3 of the lecture notes. The saddle-point of this action corresponds to the Ginzburg-Landau equations. Using the action, we can go beyond the saddle-point equations, include the fluctuations of the order parameter, and study the accuracy of the mean-field theory.

(a) Calculate the coefficient K by expanding (6.149) to second order in $\mathbf{q}$, and then doing the momentum integral and the Matsubara sum. The momentum integral is cut off at energy ω_{D} such that $T \ll \omega_{\mathrm{D}} \ll E_{\mathrm{F}}$. Within the usual approximations, the result is given by Eqs. (6.162–3).

(b) Find the saddle-point solution Δ_0 in terms of a and b by taking a functional derivative with respect to Δ and Δ^*.

(c) Expand the action to quadratic order in $\delta\Delta(\mathbf{r})=\Delta(\mathbf{r})-\Delta_0$ around the mean-field value Δ_0. The resulting action can be written in the form

$$\begin{aligned}S[\Delta,\Delta^*]=\beta V f_{\mathrm{mf}}[\Delta_0]&+\beta\int \mathrm{d}^3 r K\left(\xi_l^{-2}|\Delta_l|^2+|\partial_{\mathbf{r}}\Delta_l|^2\right)\\&+\beta\int \mathrm{d}^3 r K\left(\xi_t^{-2}|\Delta_t|^2+|\partial_{\mathbf{r}}\Delta_t|^2\right),\end{aligned} \tag{2}$$

where V is the volume of the superconductor and f_{mf} is the mean-field contribution to the free energy density. Above, we separated the longitudinal and tranverse order-parameter fluctuations by making a change of variables

$$\delta\Delta=\Delta_l+i\Delta_t, \qquad \delta\Delta^*=\Delta_l-i\Delta_t.$$

Identify the inverse coherence lengths $\xi_{l,t}^{-1}$ both below and above T_c.

(d) Determine the fluctuation contribution to the free energy of the superconductor by doing the remaining Gaussian integral. Study the singular part of the heat capacity near T_c by considering the derivative

$$C_{\mathrm{sing}}=\frac{\partial^2(\beta f)}{\partial T^2}, \tag{3}$$

where $\beta f=-\log\mathcal{Z}/V$. The mean-field theory is accurate when the mean-field discontinuity is much larger than the fluctuation contribution. When is this criterion satisfied?

2. Dzyaloshinskii-Larkin theorem $(2 + 3 + 4 + 15 + 1 = 25$ points$)$

The purpose of this exercise is to show that in the Tomonaga-Luttinger model all the loops made out of $n \leq 3$ fermionic lines vanish. This means that the RPA approximation is exact. For this exercise it is enough to consider only the right-movers and assume that the spectrum is $\xi_p = v(p - p_F)$.

(a) Let us consider a loop made out of three fermionic Green functions and with three wavy lines as external legs carrying frequencies ω_i and momenta k_i, $i = 1, 2, 3$. Physically, such a diagram represents the cubic interaction of density fluctuations (compare to polarization operator). To be precise, there are two diagrams of this type which differ by the order of wavy lines. Draw these two diagrams and write down the corresponding analytical expressions. Assume Matsubara technique for definiteness.

(b) Use the following identity

$$\frac{1}{\mathrm{i}\omega - vp} \, \frac{1}{\mathrm{i}(\omega + \nu) - v(p+q)} = \frac{1}{\mathrm{i}\nu - vq}\left[\frac{1}{\mathrm{i}\omega - vp} - \frac{1}{\mathrm{i}(\omega + \nu) - v(p+q)}\right] \qquad (4)$$

to transform the analytic expressions for the diagrams discussed in task (a). What is the graphical representation of this transformation?

(c) Show that the sum of the two diagrams from task (a) vanish.

(d) Generalize the above arguments for a loop with $n \geq 3$ fermionic lines. Note that there are now $(n-1)!$ diagrams which differ by the order of the external lines. The loop diagrams for n external momenta can be generated by taking the set of loop diagrams with $n - 1$ external momenta and inserting an extra wavy line in turn to each of the fermionic lines.

(e) Why does the line of reasoning (a)-(d) not apply to a fermionic loop made out of two fermionic lines (polarization operator)?