Karlsruher Institut für Technologie   Institute for Condensed Matter Theory
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Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 11
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1. Ginzburg-Landau action (10 + 1 + 4 + 10 = 25 points)

Consider the superconducting Ginzburg-Landau action

$$S[\Delta, \Delta^*] = \beta \int \mathrm{d}^3 r \left[a(T)|\Delta|^2 + b|\Delta|^4 + K|\partial_{\mathbf{r}}\Delta|^2\right]. \tag{1}$$

which was discussed at Sec. 6.3.3 of the lecture notes.

(a) Calculate the coefficient K from the microscopic theory.

Solution:

K is defined by the expansion

$$\chi_c^{(0)}(0, \mathbf{q}) \approx \chi_c^{(0)}(0, 0) - K|\mathbf{q}|^2, \tag{2}$$

and $\chi_c^{(0)}$ is given by

$$\chi_c^{(0)}(0, \mathbf{q}) = \frac{T}{V} \sum_{\epsilon_n, \mathbf{p}} \frac{1}{(i\epsilon_n - \xi_{\mathbf{p}})(-i\epsilon_n - \xi_{-\mathbf{p}+\mathbf{q}})}. \tag{3}$$

In a metal with large Fermi momentum, we may approximate

$$\xi_{-\mathbf{p}+\mathbf{q}} \approx \xi_{-\mathbf{p}} - \frac{\mathbf{p}\cdot\mathbf{q}}{m}, \tag{4}$$

and expand the above fraction to the second order. The first order term vanishes under the integral. We approximate $|\mathbf{p}| \approx p_{\mathrm{F}}$ in the numerator, and the density of states ν by its value at Fermi surface. For the second order term, we obtain

$$\begin{aligned} K &= -\frac{T}{q^2 m^2 V} \sum_{\epsilon_n, \mathbf{p}} \frac{(\mathbf{p}\cdot\mathbf{q})^2}{(i\epsilon_n - \xi_{\mathbf{p}})(-i\epsilon_n - \xi_{\mathbf{p}})^3} \\ &= -\frac{2T}{3m^2} \sum_{\epsilon_n} \int_{-\omega_{\mathrm{D}}}^{\omega_{\mathrm{D}}} \mathrm{d}\xi \nu \frac{1}{2} \int_{-1}^{+1} \mathrm{d}\cos\theta \frac{p_F^2 (\cos\theta)^2}{(i\epsilon_n - \xi_{\mathbf{p}})(-i\epsilon_n - \xi_{\mathbf{p}})^3} \\ &= \frac{\nu v_F^2 T}{3} \sum_{\epsilon_n} \frac{1}{\epsilon_n^3} \left(\frac{\epsilon_n \omega_{\mathrm{D}} (3\epsilon_n^2 + \omega_{\mathrm{D}}^2)}{(\epsilon_n^2 + \omega_{\mathrm{D}}^2)^2} + \tan^{-1}\left(\frac{\omega_{\mathrm{D}}}{\epsilon_n}\right)\right) \end{aligned} \tag{5}$$

When $T \ll \omega_{\mathrm{D}}$, the main contribution to the Matsubara sum comes from $|\epsilon_n| \ll \omega_{\mathrm{D}}$. At this limit, we can neglect the first term in the parentheses and approximate the arcustangent by $\pi\,\mathrm{sgn}(\epsilon_n)/2$. The sum can now be evaluated:

$$T \sum_{\epsilon_n} \frac{1}{|\epsilon_n|^3} = \frac{7\zeta(3)}{8\pi^3 T^2}. \tag{6}$$

We find the coefficient

$$K = \frac{7\pi\zeta(3)}{2\cdot 3\cdot 8\pi^3}\nu\left(\frac{v_F}{T}\right)^2 = \frac{7\zeta(3)}{48\pi^2}\nu\left(\frac{v_F}{T}\right)^2. \tag{7}$$

Since we are interested in the expansion near T_c, we may neglect the temperature dependence of K, and evaluate it at T_c:

$$K = \frac{7\zeta(3)}{48\pi^2}\nu\left(\frac{v_F}{T_c}\right)^2 = C\nu\left(\frac{v_F}{T_c}\right)^2. \tag{8}$$

(b) Find the mean-field solution Δ_0 in terms of a and b by taking a functional derivative with respect to Δ and Δ^*. Choose the phase so that Δ_0 is real.

Solution: From Eqs. (6.155) and (6.159), we have

$$a(T) = \nu t, \qquad \text{and} \qquad b = \frac{7\zeta(3)}{16\pi^2}\frac{\nu}{T_c^2} = 3C\frac{\nu}{T_c^2} \tag{9}$$

where $t = (T - T_c)/T_c$ is the temperature relative to the T_c. We are close to T_c so b can be evaluated at T_c and its temperature-dependence neglected.

The two saddle-point equations

$$\frac{1}{\beta}\frac{\delta S}{\delta\Delta} = 2a(T)\Delta^*(\mathbf{r}) + 4b\Delta^*(\mathbf{r})|\Delta(\mathbf{r})|^2 - 4K\nabla^2\Delta^*(\mathbf{r}), \tag{10}$$

$$\frac{1}{\beta}\frac{\delta S}{\delta\Delta^*} = 2a(T)\Delta(\mathbf{r}) + 4b\Delta(\mathbf{r})|\Delta(\mathbf{r})|^2 - 4K\nabla^2\Delta(\mathbf{r}) \tag{11}$$

have the uniform solution

$$|\Delta_0|^2 = -\frac{a(T)}{2b}, \tag{12}$$

if the coefficient of the quadratic term is negative ($a(T) < 0$) and quartic term is positive ($b > 0$). This is satisfied when $T < T_c$. Above T_c, there is only the trivial solution $\Delta_0 = 0$. The phase of Δ_0 is not fixed by the saddle-point equations. We choose the phase so that Δ_0 is a positive real number.

(c) Expand the action to quadratic order in $\delta\Delta(\mathbf{r}) = \Delta(\mathbf{r}) - \Delta_0$ around the mean-field value Δ_0. The resulting action can be written in the form

$$\begin{aligned} S[\Delta,\Delta^*] = \beta V f_{\mathrm{mf}}[\Delta_0] &+ \beta\int \mathrm{d}^3r K\left(\xi_l^{-2}|\Delta_l|^2 + |\partial_{\mathbf{r}}\Delta_l|^2\right) \\ &+ \beta\int \mathrm{d}^3r K\left(\xi_t^{-2}|\Delta_t|^2 + |\partial_{\mathbf{r}}\Delta_t|^2\right), \end{aligned} \tag{13}$$

where V is the volume of the superconductor and f_{mf} is the mean-field contribution to the free energy density. Above, we separated the longitudinal and tranverse order-parameter fluctuations:

$$\delta\Delta = \Delta_l + i\Delta_t, \qquad \delta\Delta^* = \Delta_l - i\Delta_t.$$

Identify the inverse coherence lengths $\xi_{l,t}^{-1}$ both below and above T_c.

Solution:

Above T_c, there is no difference between the longitudinal and transverse directions. Expanding the action around $\Delta = 0$ to quadratic order simply correspond to throwing away the quartic b-term. We find

$$\xi_{l,t}^{-2} = \frac{a(T)}{K} = \frac{\nu t}{K} = \frac{|t| T_c^2}{C v_F^2}, \tag{14}$$

where $t = (T - T_c)/T_c$.
Below T_c, we find

$$\begin{aligned}
S[\Delta, \Delta^*] &= \beta \int_x \Big\{ a(T)(\Delta_0^2 + 2\Delta_0 \Delta_l + \Delta_l^2 + \Delta_t^2) \\
&\quad + b(\Delta_0^2 + 2\Delta_0\Delta_l + \Delta_l^2 + \Delta_t^2)^2 \\
&\quad + K[(\partial_{\mathbf{r}}\Delta_l)^2 + (\partial_{\mathbf{r}}\Delta_t)^2] \Big\} \\
&\approx \beta \int_x \Big\{ a(T)(\Delta_0^2 + 2\Delta_0 \Delta_l + \Delta_l^2 + \Delta_t^2) \\
&\quad + b(\Delta_0^4 + 4\Delta_0^3\Delta_l + 6\Delta_0^2\Delta_l^2 + 2\Delta_0^2\Delta_t^2) \\
&\quad + K[(\partial_{\mathbf{r}}\Delta_l)^2 + (\partial_{\mathbf{r}}\Delta_t)^2] \Big\} \\
&= \beta \int_x \Big\{ a(T)(\Delta_0^2 + \Delta_l^2 + \Delta_t^2) + b(\Delta_0^4 + 6\Delta_0^2\Delta_l^2 + 2\Delta_0^2\Delta_t^2) \\
&\quad + K[(\partial_{\mathbf{r}}\Delta_l)^2 + (\partial_{\mathbf{r}}\Delta_t)^2] \Big\} \\
&= \beta V \underbrace{(a(T)\Delta_0^2 + b\Delta_0^4)}_{f_{\mathrm{mf}}[\Delta_0]} + \beta \int_x \Big\{ \underbrace{[a(T) + 2b\Delta_0^2]}_{0} \Delta_t^2 + K(\partial_{\mathbf{r}}\Delta_t)^2 \Big\} \\
&\quad + \beta \int_x \Big\{ \underbrace{[a(T) + 6b\Delta_0^2]}_{-2a(T)} \Delta_l^2 + K(\partial_{\mathbf{r}}\Delta_l)^2 \Big\}
\end{aligned} \tag{15}$$

where the linear-in-Δ_l terms canceled because Δ_0 satisfies the saddle-point equation $a(T) = -2b\Delta_0^2$. The coherence lenghts are

$$\xi_t^{-2} = 0, \tag{16}$$

$$\xi_l^{-2} = -\frac{2a(T)}{K} = \frac{2\nu|t|}{K}. \tag{17}$$

The vanishing ξ_t^{-1} is related to the fact that the transverse mode is the Goldstone mode of the superconductor.

(d) Determine the fluctuation contribution to the free energy of the superconductor by doing the remaining Gaussian integral. Study the singular part of the heat capacity near T_c by considering the derivative

$$C_{\mathrm{sing}} = \frac{\partial^2(\beta f)}{\partial T^2}, \tag{18}$$

where $\beta f = -\log \mathcal{Z}/V$. The mean-field theory is accurate when the mean-field discontinuity is much larger than the fluctuation contribution. When is this criterion satisfied?

Solution:
To calculate the path integral, we first express the action in momentum space. The Fourier transform is

$$\Delta_{l,t}(\mathbf{r}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \Delta_{l,t}(\mathbf{q}) \tag{19}$$

The squares and the squares of derivatives transform as

$$\begin{aligned}
\int_x \Delta_i(\mathbf{r})^2 &= \int_x \int_q \int_{q'} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}} \Delta_i(\mathbf{q})\Delta_i(\mathbf{q}') \\
&= \int_q \Delta_i(\mathbf{q})\Delta_i(-\mathbf{q}) = \int_q |\Delta_i(\mathbf{q})|^2, \\
\int_x [\partial_{\mathbf{r}}\Delta_i(\mathbf{r})]^2 &= \int_x \int_q \int_{q'} (i\mathbf{q})\cdot(i\mathbf{q}') e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}} \Delta_i(\mathbf{q})\Delta_i(\mathbf{q}') \\
&= \int_q q^2 \Delta_i(\mathbf{q})\Delta_i(-\mathbf{q}) = \int_q q^2 |\Delta_i(\mathbf{q})|^2,
\end{aligned} \tag{20}$$

since for real field $\Delta_i(-\mathbf{q}) = \Delta_i(\mathbf{q})^*$.
The action becomes

$$\begin{aligned}
S_i[\Delta_i] &= \beta \int_x K(\xi_i^{-2}\Delta_i^2 + (\partial_{\mathbf{r}}\Delta)^2) \\
&= \beta \int_q \Delta_i(-\mathbf{q}) K(\xi_i^{-2} + q^2)\Delta_i(\mathbf{q})
\end{aligned} \tag{21}$$

The path integral can be written as a product of three terms:

$$\mathcal{Z} = e^{-\beta V f_{\mathrm{mf}}[\Delta_0]} \int \mathcal{D}\Delta_l \exp(-S_l[\Delta_l]) \int \mathcal{D}\Delta_t \exp(-S_t[\Delta_t]). \tag{22}$$

Evaluating the fluctuation terms, we get

$$\begin{aligned}
\int \mathcal{D}\Delta_i \exp(-S_i[\Delta_i]) &= \prod_{\mathbf{q}} \int \mathrm{d}\Delta_{\mathbf{q}} \mathrm{d}\Delta_{\mathbf{q}}^* \exp\big(-\Delta_{\mathbf{q}}^*[\beta K(\xi_q^{-2} + q^2)]\Delta_{\mathbf{q}}\big) \\
&= \prod_{\mathbf{q}} \frac{\pi}{\beta K(\xi^{-2} + q^2)} \propto \exp\left(-\sum_{\mathbf{q}} \log[K(\xi^{-2} + q^2)]\right)
\end{aligned} \tag{23}$$

The free energy density is

$$\beta \Delta f = -\frac{\log \mathcal{Z}}{V} = \beta f_{\mathrm{mf}}[\Delta_0] + \int_{\mathbf{q}} \log[K(\xi_l^{-2} + q^2)] + \int_{\mathbf{q}} \log[K(\xi_t^{-2} + q^2)]. \tag{24}$$

Let us then consider the divergence of the heat capacity near the T_c. Heat capacity itself is the first derivative of energy, but to study its divergence, we calculate the second derivative. The temperature dependence of the mean-field part is

$$f_{\mathrm{mf}}[\Delta_0] = a\Delta_0^2 + b\Delta_0^4 = \begin{cases} -\frac{a^2}{4b} = -\frac{\nu(T_c t)^2}{12C}. & t < 0 \\ 0, & t > 0 \end{cases} \tag{25}$$

There is a discontinuity in the second derivative at critical temperature $t = 0$:

$$\beta \frac{\partial^2 f_{\mathrm{mf}}}{\partial t^2} = \begin{cases} -\frac{\nu T_C}{6C}, & t > 0 \\ 0. & t > 0 \end{cases} \tag{26}$$

The temperature dependence of the fluctuation part comes from the correlation lengths. Below T_c, only the longitudinal fluctuations depend on temperature, and we have

$$-\sum_i \frac{\partial^2(\beta f_i)}{\partial t^2} = \int_q \frac{4\nu^2}{[-2\nu t + Kq^2]^2} \tag{27}$$

Above T_c, both modes depend on temperature and we get

$$\begin{aligned} -\sum_i \frac{\partial^2(\beta f_i)}{\partial t^2} &= -2\partial_t \int_q \frac{\partial_t(\nu t)}{\nu t + Kq^2} = 2\int_q \frac{\nu^2}{[\nu t + Kq^2]^2} \\ &= \frac{1}{4\pi C^{3/2}} \left(\frac{T_C}{v_F}\right)^3 t^{-1/2} \end{aligned} \tag{28}$$

Below T_c, only the transverse mode depends on temperature:

$$\begin{aligned} \sum_i \frac{\partial^2(\beta f_i)}{\partial t^2} &= \partial_t \int_q \frac{\partial_t(2\nu|t|)}{2\nu|t| + (Kq)^2} = -\int_q \frac{4\nu^2}{[2\nu|t| + (Kq)^2]^2} \\ &= \frac{1}{2^{3/2}\pi C^{3/2}} \left(\frac{T_C}{v_F}\right)^3 t^{-1/2} \end{aligned} \tag{29}$$

Now we may compare with the divergence:

$$\begin{aligned} R &\sim \frac{1}{4\pi C^{3/2}} \left(\frac{T_C}{v_F}\right)^3 \cdot \frac{6C}{\nu T_C} \cdot t^{-1/2} \\ &\sim \frac{3}{2\pi C^{1/2}} \left(\frac{T_C^2}{\nu v_F^3}\right) t^{-1/2}, \\ &\sim \frac{3\pi}{4C^{1/2}} \left(\frac{T_C}{E_F}\right)^2 t^{-1/2} \end{aligned} \tag{30}$$

where the prefactor is somewhat different below and above T_C, but the exponent is 1/2 in both cases. One might now substitute some material parameters into this expression. At the last step, we have used the properties of the Fermi gas to extract the Fermi energy. At weak coupling $T_C \ll E_F$, we find that the approximation is very good.

What we have done here does not go beyond mean-field theory. We have calculated the fluctuations based on mean-field theory, and estimated the validity of that approximation from within the theory. The exponent 1/2 is thus a mean-field prediction, and we would expect a more careful calculation (renormalization group) to change it at the regime where the fluctuations are significant. However such regime only occurs very close to T_C, and within the experimentally accessible range in many superconductors the exponent of the heat capacity agrees with the mean-field prediction.

2. Dzyaloshinskii-Larkin theorem $(2+3+4+15+1=25 \text{ points})$

The purpose of this exercise is to show that in the Tomonaga-Luttinger model all the loops made out of $n \leq 3$ fermionic lines vanish. This means that the RPA approximation is exact.

(a) Let us consider a loop made out of three fermionic Green functions and with three wavy lines as external legs carrying frequencies ω_i and momenta k_i, $i = 1, 2, 3$. Physically, such a diagram represents the cubic interaction of density fluctuations (compare to polarization operator). To be precise, there are two diagrams of this type which differ by the order of wavy lines. Draw these two diagrams and write down the corresponding analytical expressions. Assume Matsubara technique for definiteness.

Solution:

Let us collect the momenta and frequencies to a 4-vector $p_i = (\mathbf{k}_i, \omega_i)$. The third order diagrams are illustrated in Fig. X. Using Feynman rules, we can write down the expressions associated with them:

$$S_1 = -\sum_{p_0} G(p_0)G(p_0+p_1)G(p_0+p_1+p_2), \tag{31}$$

$$S_2 = -\sum_{p_0} G(p_0)G(p_0+p_2)G(p_0+p_1+p_2), \tag{32}$$

(b) Use the following identity

$$G(p)G(p+p_n) = \frac{G(p) - G(p+p_n)}{\mathrm{i}\nu_n - vq_n} \tag{33}$$

to transform the analytic expressions for the diagrams discussed in task (a). What is the graphical representation of this transformation?

Solution:

To get the same prefactor, we always apply the transformation to the term $G(x)G(x+p_2)$ in the product:

$$S_1 = -\frac{1}{\mathrm{i}\nu_2 - vq_2}\sum_{p_0}\left[G(p_0)G(p_0+p_1) - G(p_0)G(p_0+p_1+p_2)\right] \tag{34}$$

$$S_2 = -\frac{1}{\mathrm{i}\nu_2 - vq_2}\sum_{p_0}\left[G(p_0)G(p_0+p_1+p_2) - G(p_0+p_2)G(p_0+p_1+p_2)\right] \tag{35}$$

(c) Show that the sum of the two diagrams from task (a) vanish.

Solution:

$$\begin{aligned} S_1 + S_2 = &-\frac{1}{\mathrm{i}\nu_2 - vq_2}\sum_{p_0}\Big[G(p_0)G(p_0+p_1) - \cancel{G(p_0)G(p_0+p_1+p_2)}\Big] \qquad (36)\\ &-\frac{1}{\mathrm{i}\nu_2 - vq_2}\sum_{p_0}\Big[\cancel{G(p_0)G(p_0+p_1+p_2)} - G(p_0+p_1)G(p_0+p_1+p_2)\Big] \\ =& -\frac{1}{\mathrm{i}\nu_2 - vq_2}\sum_{p_0}\left[G(p_0)G(p_0+p_1) - G(p_0+p_2)G(p_0+p_1+p_2)\right] \qquad (37)\\ =& \, 0, \qquad (38)\end{aligned}$$

where on the last line we shifted the integration variable in the second term to make the integrand vanish.

(d) Generalize the above arguments to the case of arbitrary fermionic loop with more than 2 fermionic lines.

Solution:

Let us consider the case with $n+1$ fermionic lines and $n+1$ external fields. We define the function

$$\gamma^{(n+1)}(p_0; p_1, p_2, \ldots, p_n) = G(p_0)G(p_0+p_1)\cdots G(p_0+p_1+\cdots+p_n), \tag{39}$$

which includes $n+1$ fermionic propagators and $p_j = (\omega_j, k_j)$ are the external energy-momenta.

We only need to consider a subset of diagrams. We choose a permutation $p_{i_1}, p_{i_2}, \ldots, p_{i_{n-1}}$ of the first $n-1$ external momenta. Then we form the sum of the n diagrams in which p_n is placed after p_{i_j} for each $j = 0, \ldots, n-1$. This gives us the sum

$$\begin{aligned} S_{i_1,\ldots,i_{n-1}}(p_0) =& \gamma^{(n+1)}(p_0; p_n, p_{i_1}, p_{i_2}, \ldots, p_{i_{n-1}}) \\ &+\gamma^{(n+1)}(p_0; p_{i_1}, p_n, p_{i_2}, \ldots, p_{i_{n-1}}) \\ &+\gamma^{(n+1)}(p_0; p_{i_1}, p_{i_2}, p_n, \ldots, p_{i_{n-1}}) \\ &\vdots \\ &+\gamma^{(n+1)}(p_0; p_{i_1}, p_{i_2}, \ldots, p_n, p_{i_{n-1}}) \\ &+\gamma^{(n+1)}(p_0; p_{i_1}, p_{i_2}, \ldots, p_{i_{n-1}}, p_n). \end{aligned} \tag{40}$$

We then use the identity

$$G(p)G(p+p_n) = \frac{G(p) - G(p+p_n)}{\mathrm{i}\omega_n - vk_n} \tag{41}$$

for each γ. For $1 \le j < n-1$ it gives

$$\begin{aligned} \gamma^{(n+1)}(p_0; \ldots, p_{i_j}, p_n, p_{i_{j+1}}, \ldots) = \frac{1}{\mathrm{i}\omega_n - vk_n}\Big[&\gamma^{(n)}(p_0; \ldots, p_{i_j}, p_{i_{j+1}}+p_n, \ldots) \\ &-\gamma^{(n)}(p_0; \ldots, p_{i_j}+p_n, p_{i_{j+1}}, \ldots)\Big] \end{aligned} \tag{42}$$

When p_n is the first one, we have

$$\begin{aligned} \gamma^{(n)}(p_0; p_n, p_{i_1}, \ldots, p_{i_{n-1}}) = \frac{1}{\mathrm{i}\omega_n - vk_n}\Big[&\gamma^{(n-1)}(p_0; p_{i_1}+p_n, \ldots, p_{i_{n-1}}) \\ &-\gamma^{(n-1)}(p_0+p_n; p_{i_1}, \ldots, p_{i_{n-1}})\Big], \end{aligned} \tag{43}$$

Now the sum becomes

$$\begin{aligned} &-(\mathrm{i}\omega_n - vk_n)S^{(n)}_{i_1,\ldots,i_{n-1}}(p_0) = \\ &\quad \gamma^{(n-1)}(p_0+p_n; p_{i_1}, p_{i_2}, \ldots, p_{i_{n-1}}) - \gamma^{(n-1)}(p_0; p_{i_1}+p_n, p_{i_2}, \ldots, p_{i_{n-1}}) \\ &\quad + \gamma^{(n-1)}(p_0; p_{i_1}+p_n, p_{i_2}, \ldots, p_{i_{n-1}}) - \gamma^{(n-1)}(p_0; p_{i_1}, p_{i_2}+p_n, \ldots, p_{i_{n-1}}) \\ &\quad \vdots \\ &\quad + \gamma^{(n-1)}(p_0; p_{i_1}, p_{i_2}, \ldots, p_{i_{n-2}}+p_n, p_{i_{n-1}}) - \gamma^{(n-1)}(p_0; p_{i_1}, p_{i_2}, \ldots, p_{i_{n-1}}+p_n) \\ &\quad + \gamma^{(n-1)}(p_0; p_{i_1}, p_{i_2}, \ldots, p_{i_{n-1}}+p_n) - \gamma^{(n-1)}(p_0; p_{i_1}, p_{i_2}, \ldots, p_{i_{n-1}}). \end{aligned} \tag{44}$$

Like with $n+1=3$ case, there is a telescoping effect, as the adjacent terms cancel each other and we left with only the first and the last term

$$-(\mathrm{i}\omega_n - vk_n)S^{(n)}_{i_1,\ldots,i_{n-1}}(p_0) = \gamma^{(n-1)}(p_0+p_n;p_{i_1},p_{i_2},\ldots,p_{i_{n-1}}) - \gamma^{(n-1)}(p_0;p_{i_1},p_{i_2},\ldots,p_{i_{n-1}}). \tag{45}$$

Now we shift $p_0+p_n \to p_0$. After this shift the sum-integral over $S^{(n)}$ vanishes.

(e) Why does the line of reasoning (a)-(c) not apply to a fermionic loop made out of two fermionic lines (polarization operator)?

Solution: For two fermionic lines the expression is

$$-\sum_{p_0} G(p_0)G(p_0+p_1) = \frac{1}{i\omega_1 - \alpha v k_1}\sum_{p_0}\left[G(p_0) - G(p_0+p_1)\right] \tag{46}$$

and it seems like we can just shift the integration variable to make the two propagators cancel against each other. However, the integrals are now divergent and we need to add a cutoff to make them converge. The presence of a cutoff forbids us from shifting the integration variable.

This problem does not arise for higher order diagrams as they have a higher power of p in the denominator.