Karlsruher Institut für Technologie   Institute for Condensed Matter Theory
Institute for Quantum Materials and Technologies

Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 12
Deadline: 21 July 2023

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1. Fermion operators in bosonization: (15 + 15 points)

In Sec. 7.6 of the Lecture Notes, the bosonized representation of fermion operators was introduced:

$$\psi_\alpha^\dagger(x) = A\, U_\alpha^\dagger \exp\{-\mathrm{i}\alpha k_\mathrm{F} x - \mathrm{i}[\theta(x) - \alpha\phi(x)]\}\,, \qquad (1)$$

$$\psi_\alpha(x) = A\, U_\alpha \exp\{\mathrm{i}\alpha k_\mathrm{F} x + \mathrm{i}[\theta(x) - \alpha\phi(x)]\}\,. \qquad (2)$$

(a) Using the commutation relations for the bosonic fields $\theta(x)$ and $\phi(x)$, as well as the anticommutation for Klein factors U_α, derive the anticommutation relations for the fermion operators on the same ($\alpha = \alpha'$) and different($\alpha \neq \alpha'$) branches, and determine the normalization constant A. The Baker-Campbell-Hausdorff formula for operators B and C can be used here: if $D = [B, C]$ satisfies $[B, D] = [C, D] = 0$, then $\mathrm{e}^B \mathrm{e}^C = \mathrm{e}^{B+C}\mathrm{e}^{D/2}$

(b) Substituting the bosonized representation of the fermionic operators into the fermionic form of the free Hamiltonian for right movers with the linear dispersion,

$$\hat{H}_{0,+} = v_F \int dx\, \psi_+^\dagger(x) \left(-\mathrm{i}\frac{\partial}{\partial x} - k_\mathrm{F}\right) \psi_+(x),$$

derive the bosonized form of $\hat{H}_{0,+}$. For this purpose, use the “point-splitting” procedure by first replacing $\psi_+^\dagger(x)$ with $\psi_+^\dagger(x + d)$, expanding the fields in small d, and taking the limit $d \to 0$ at the end of the calculation, with the ground-state average removed as in the normal-ordered product.

2. Zero-bias anomaly in the Luttinger liquid (20 points)

In Sec. 7.7.5 of the Lecture Notes, the energy dependence of the tunneling density of states in a Luttinger liquid was found: $\nu(\epsilon) \propto |\epsilon|^\gamma$, with exponent $\gamma = (K + K^{-1})/2 - 1$ determined by the Luttinger parameter K. Starting with the expression for the Green's function in the spinless case, express $\mathrm{Im}\, G^\mathrm{R}(0, t)$ through $G^>$ and $G^<$, and derive the full expression for $\nu(\epsilon)$ at zero temperature, including the numerical factor in front of the power-law energy dependence. The following identities for the Gamma function $\Gamma(x)$ can be useful: $\Gamma(x) = \int_0^\infty \mathrm{d}s\, \exp(-s) s^{x-1}$ and $\Gamma(x)\Gamma(1 - x) = \pi/\sin(\pi x)$.