Karlsruher Institut für Technologie   Institute for Condensed Matter Theory
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Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 12
Deadline: 21 July 2023

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1. Fermion operators in bosonization: (15 + 15 points)

In Sec. 7.6 of the Lecture Notes, the bosonized representation of fermion operators was introduced:

$$\psi^\dagger_\alpha(x) = A\, U^\dagger_\alpha \exp\{-\mathrm{i}\alpha k_{\mathrm F} x - \mathrm{i}[\theta(x) - \alpha\phi(x)]\}\,, \tag{1}$$
$$\psi_\alpha(x) = A\, U_\alpha \exp\{\mathrm{i}\alpha k_{\mathrm F} x + \mathrm{i}[\theta(x) - \alpha\phi(x)]\}\,. \tag{2}$$

(a) Using the commutation relations for the bosonic fields $\theta(x)$ and $\phi(x)$, as well as the anticommutation for Klein factors U_α, derive the anticommutation relations for the fermion operators on the same ($\alpha = \alpha'$) and different($\alpha \neq \alpha'$) branches, and determine the normalization constant A. The Baker-Campbell-Hausdorff formula for operators B and C can be used here: if $D = [B, C]$ satisfies $[B, D] = [C, D] = 0$, then $\mathrm{e}^B\mathrm{e}^C = \mathrm{e}^{B+C}\mathrm{e}^{D/2}$

Solution: We need to calculate

$$\{\psi^\dagger_\alpha(x), \psi_\beta(x')\} \tag{3}$$
$$\{\psi_\alpha(x), \psi_\beta(x')\}. \tag{4}$$

We know from the lecture notes, Eqs. (7.100), (7.101):

$$U_\alpha U^\dagger_\alpha = U^\dagger_\alpha U_\alpha = 1 \tag{5}$$
$$\{U_\alpha, U_{\alpha'}\} = \{U^\dagger_\alpha, U^\dagger_{\alpha'}\} = \{U^\dagger_\alpha, U_{\alpha'}\} = 0 \quad \text{for } \alpha \neq \alpha' \tag{6}$$

From the definitions of $\theta(x)$ and $\phi(x)$ (Eq. (7.104))

$$\theta(x) = \frac{\mathrm{i}\pi}{L}\sum_{q\neq 0}\frac{1}{q}\mathrm{e}^{-\mathrm{i}qx}[-\varrho_+(q) - \varrho_-(q)]\mathrm{e}^{-|q|\lambda/2} \tag{7}$$

$$\phi(x) = \frac{\mathrm{i}\pi}{L}\sum_{q\neq 0}\frac{1}{q}\mathrm{e}^{-\mathrm{i}qx}[\varrho_+(q) - \varrho_-(q)]\mathrm{e}^{-|q|\lambda/2} \tag{8}$$

$$\varrho_+(q) = \left(\frac{|q|L}{2\pi}\right)^{\frac12}\begin{cases} b^\dagger_{+,|q|} & q > 0 \\ b_{+,|q|} & q < 0\end{cases} \tag{9}$$

$$\varrho_-(q) = \left(\frac{|q|L}{2\pi}\right)^{\frac12}\begin{cases} b_{-,-|q|} & q > 0 \\ b^\dagger_{-,-|q|} & q < 0\end{cases} \tag{10}$$

$$[b^\dagger_{\alpha,q}, b_{\alpha',q'}] = -\delta_{\alpha,\alpha'}\delta_{q,q'} \tag{11}$$

It is convenient to introduce new fields that appear in the exponents of the fermionic operators:

$$\zeta_\alpha(x) := \theta(x) - \alpha\phi(x) \tag{12}$$
$$= \frac{\mathrm{i}\pi}{L}\sum_{q\neq 0}\frac{1}{q}\mathrm{e}^{-\mathrm{i}qx}\mathrm{e}^{-|q|\lambda/2}[\varrho_+(q) - \varrho_-(q) - \alpha[-\varrho_+(q) - \varrho_-(q)]] \tag{13}$$
$$= \frac{\mathrm{i}\pi}{L}\sum_{q\neq 0}\frac{1}{q}\mathrm{e}^{-\mathrm{i}qx}\mathrm{e}^{-|q|\lambda/2}[\varrho_+(q) - \varrho_-(q) - \alpha[-\varrho_+(q) - \varrho_-(q)]] \tag{14}$$
$$= \alpha\frac{2\mathrm{i}\pi}{L}\sum_{q\neq 0}\frac{1}{q}\mathrm{e}^{-\mathrm{i}qx}\mathrm{e}^{-|q|\lambda/2}\varrho_\alpha(q) \tag{15}$$

Consider + operator:

$$\zeta_+(x) = \frac{2\mathrm{i}\pi}{L}\sum_{q\neq 0}\frac{1}{q}\mathrm{e}^{-\mathrm{i}qx}\mathrm{e}^{-|q|\lambda/2}\varrho_+(q) \tag{16}$$
$$= \frac{2\mathrm{i}\pi}{L}\sum_{q>0}\frac{1}{q}\mathrm{e}^{-|q|\lambda/2}\left[\varrho_+(q)\mathrm{e}^{-\mathrm{i}qx} - \varrho_+(-q)\mathrm{e}^{\mathrm{i}qx}\right] \tag{17}$$
$$= \mathrm{i}\sqrt{\frac{2\pi}{L}}\sum_{q>0}\frac{\mathrm{e}^{-|q|\lambda/2}}{\sqrt{q}}\left[b^\dagger_{+,q}\mathrm{e}^{-\mathrm{i}qx} - b_{+,q}\mathrm{e}^{\mathrm{i}qx}\right] \tag{18}$$
$$:= \eta_+(x) + \eta^\dagger_+(x) \tag{19}$$

where

$$\eta^\dagger_+(x) = \mathrm{i}\sqrt{\frac{2\pi}{L}}\sum_{q>0}\frac{\mathrm{e}^{-|q|\lambda/2}}{\sqrt{q}}b^\dagger_{+,q}\mathrm{e}^{-\mathrm{i}qx}. \tag{20}$$

Commutation relations for these new operators:

$$[\eta_+(x), \eta_+(x')] = [\eta^\dagger_+(x), \eta^\dagger_+(x')] = 0 \tag{21}$$

because $[b_q, b_{q'}] = [b^\dagger_q, b^\dagger_{q'}] = 0$. Furthermore:

$$[\eta_+(x), \eta^\dagger_+(x')] = \frac{2\pi}{L}\sum_{q,q'>0}\frac{\mathrm{e}^{-(q+q')\lambda/2}}{\sqrt{qq'}}\mathrm{e}^{\mathrm{i}qx}\mathrm{e}^{-\mathrm{i}q'x'}\delta_{q,q'} \tag{22}$$
$$= \frac{2\pi}{L}\sum_{q>0}\frac{\mathrm{e}^{-q\lambda}}{q}\mathrm{e}^{\mathrm{i}q(x-x')} \quad q = 2\pi j/L,\ j\in\mathbb{Z} \tag{23}$$
$$= \sum_{j=1}^{\infty}\frac{\mathrm{e}^{-\frac{2\pi j}{L}[\mathrm{i}(x'-x)+\lambda]}}{q} \tag{24}$$
$$= -\log\left[1 - \mathrm{e}^{\frac{2\pi}{L}[\mathrm{i}(x-x')-\lambda]}\right] \tag{25}$$
$$\overset{L\to\infty}{\to} -\log\left[\frac{2\pi}{L}[\lambda - \mathrm{i}(x-x')]\right] \tag{26}$$

The sum can be evaluated with Mathematica.
We use the new operators to rewrite the fermionic creation and annihilation opera-

tors:

$$\psi_+(x) = AU_+ e^{ik_F x} \exp(i[\theta(x) - \phi(x)]) \tag{27}$$
$$= AU_+ e^{ik_F x} \exp(i\zeta_+(x)) \tag{28}$$
$$= AU_+ e^{ik_F x} \exp\Big(i[\eta_+(x) + \eta_+^\dagger(x)]\Big) \tag{29}$$

Now we use the BCH formula with $B = i\eta_+(x)$, $C = i\eta_+^\dagger(x)$. The commutator of these operators is a number, thus D commutes with B and C as required. We find

$$\psi_+(x) = AU_+ e^{ik_F x} e^{i\eta_+^\dagger(x)} e^{i\eta_+(x)} \exp(\log(2\pi\lambda/L)) \tag{30}$$
$$= \sqrt{\frac{2\pi\lambda}{L}} AU_+ e^{ik_F x} e^{i\eta_+^\dagger(x)} e^{i\eta_+(x)} \tag{31}$$

and similarly

$$\psi_+^\dagger(x) = \sqrt{\frac{2\pi\lambda}{L}} AU_+^\dagger e^{-ik_F x} e^{-i\eta_+^\dagger(x)} e^{-i\eta_+(x)}. \tag{32}$$

Thus:

$$\psi_+(x)\psi_+^\dagger(x') = \frac{2\pi\lambda}{L} A^2 U_+ e^{ik_F(x-x')} e^{i\eta_+^\dagger(x)} e^{i\eta_+(x)} U_+^\dagger e^{-i\eta_+^\dagger(x')} e^{-i\eta_+(x')} \tag{33}$$
$$= A^2 \frac{2\pi\lambda}{L} U_+ U_+^\dagger e^{ik_F(x-x')} e^{i\eta_+^\dagger(x)} e^{-i\eta_+^\dagger(x')} e^{i\eta_+(x)} e^{-i\eta_+(x')} e^{-[\eta_+^\dagger(x'),\eta_+(x)]} \tag{34}$$
$$= A^2 \frac{2\pi\lambda}{L} e^{ik_F(x-x')} e^{i(\eta_+^\dagger(x)-\eta_+^\dagger(x'))} e^{i(\eta_+(x)-\eta_+(x'))} \left[\frac{2\pi}{L}[\lambda - i(x-x')]\right]^{-1} \tag{35}$$
$$\psi_+^\dagger(x')\psi_+(x) = \frac{2\pi\lambda}{L} A^2 e^{ik_F(x-x')} e^{-i\eta_+^\dagger(x')} e^{-i\eta_+(x')} e^{i\eta_+(x)} e^{i\eta_+^\dagger(x)} \tag{36}$$
$$= A^2 \frac{2\pi\lambda}{L} e^{ik_F(x-x')} e^{i(\eta_+^\dagger(x)-\eta_+^\dagger(x'))} e^{i(\eta_+(x)-\eta_+(x'))} \left[\frac{2\pi}{L}[\lambda - i(x'-x)]\right]^{-1} \tag{37}$$

where we used $U_+^\dagger U_+ = U_+ U_+^\dagger = 1$, and BCH to commute the exponentials:

$$e^B e^C = e^{B+C} e^{D/2} \tag{38}$$
$$= e^C e^B e^D. \tag{39}$$

We find the anticommutator

$$\{\psi_+(x), \psi_+^\dagger(x')\} = A^2 \frac{2\pi\lambda}{L} e^{ik_F(x-x')} e^{i(\eta_+^\dagger(x)-\eta_+^\dagger(x'))} e^{i(\eta_+(x)-\eta_+(x'))} \times \tag{40}$$
$$\times \left[\left[\frac{2\pi}{L}[\lambda - i(x-x')]\right]^{-1} + \left[\frac{2\pi}{L}[\lambda - i(x'-x)]\right]^{-1}\right] \tag{41}$$
$$\overset{\lambda\to 0}{\to} A^2 \frac{2\pi\lambda}{L} e^{ik_F(x-x')} e^{i(\eta_+^\dagger(x)-\eta_+^\dagger(x'))} e^{i(\eta_+(x)-\eta_+(x'))} L\delta(x-x') \tag{42}$$
$$= A^2 2\pi\lambda\delta(x-x') \tag{43}$$

We obtain the desired fermionic anti-commutation relation, by setting $A := (2\pi\lambda)^{-1}$. We repeat the procedure for the other same-branch anti-commutators:

$$\psi_+(x)\psi_+(x') = \frac{1}{2\pi\lambda}\frac{2\pi\lambda}{L}U_+^2 \mathrm{e}^{\mathrm{i}k_\mathrm{F}(x+x')}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\mathrm{e}^{\mathrm{i}\eta_+(x)}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x')}\mathrm{e}^{\mathrm{i}\eta_+(x')} \tag{44}$$

$$= \frac{U_+^2 \mathrm{e}^{\mathrm{i}k_\mathrm{F}(x+x')}}{L}\mathrm{e}^{\mathrm{i}(\eta_+^\dagger(x)+\eta_+^\dagger(x'))}\mathrm{e}^{\mathrm{i}(\eta_+(x)+\eta_+(x'))}\mathrm{e}^{[\eta_+^\dagger(x'),\eta_+(x)]} \tag{45}$$

$$= \frac{U_+^2 \mathrm{e}^{\mathrm{i}k_\mathrm{F}(x+x')}}{L}\mathrm{e}^{\mathrm{i}(\eta_+^\dagger(x)+\eta_+^\dagger(x'))}\mathrm{e}^{\mathrm{i}(\eta_+(x)+\eta_+(x'))}\frac{2\pi}{L}\left[\lambda - \mathrm{i}(x-x')\right] \tag{46}$$

The anti-commutator:

$$\{\psi_+(x), \psi_+(x')\} = \frac{2\pi U_+^2 \mathrm{e}^{\mathrm{i}k_\mathrm{F}(x+x')}}{L^2}\mathrm{e}^{\mathrm{i}(\eta_+^\dagger(x)+\eta_+^\dagger(x'))}\mathrm{e}^{\mathrm{i}(\eta_+(x)+\eta_+(x'))}\left([\lambda - \mathrm{i}(x'-x)] + [\lambda - \mathrm{i}(x-x')]\right) \tag{47}$$

$$= \frac{4\pi U_+^2 \mathrm{e}^{\mathrm{i}k_\mathrm{F}(x+x')}}{L^2}\mathrm{e}^{\mathrm{i}(\eta_+^\dagger(x)+\eta_+^\dagger(x'))}\mathrm{e}^{\mathrm{i}(\eta_+(x)+\eta_+(x'))}\lambda \overset{\lambda\to 0}{\to} 0. \tag{48}$$

Same result for

$$\{\psi_+^\dagger(x), \psi_+^\dagger(x')\} = \{\psi_+(x), \psi_+(x')\}^\dagger. \tag{49}$$

The $-$-branch:

$$\zeta_-(x) = -\frac{2\mathrm{i}\pi}{L}\sum_{q\neq 0}\frac{1}{q}\mathrm{e}^{-\mathrm{i}qx}\mathrm{e}^{-|q|\lambda/2}\varrho_-(q) \tag{50}$$

$$= -\frac{2\mathrm{i}\pi}{L}\sum_{q>0}\frac{1}{q}\mathrm{e}^{-|q|\lambda/2}\left[\varrho_-(q)\mathrm{e}^{-\mathrm{i}qx} - \varrho_-(-q)\mathrm{e}^{\mathrm{i}qx}\right] \tag{51}$$

$$= -\mathrm{i}\sqrt{\frac{2\pi}{L}}\sum_{q>0}\frac{1}{\sqrt{q}}\mathrm{e}^{-|q|\lambda/2}\left[b_{-,-q}\mathrm{e}^{-\mathrm{i}qx} - b_{-,-q}^\dagger\mathrm{e}^{\mathrm{i}qx}\right] \tag{52}$$

$$:= \eta_-(x) + \eta_-^\dagger(x) \tag{53}$$

$$\{\psi_+^\dagger(x), \psi_-(x')\} = A^2 U_+^\dagger U_- \mathrm{e}^{-\mathrm{i}k_\mathrm{F}(x+x')}\left[\exp(-\mathrm{i}\zeta_+(x)), \exp(\mathrm{i}\zeta_-(x))\right] = 0 \tag{54}$$

Here, we used $\{U_+^\dagger, U_-\} = 0$. The commutator vanishes, because ζ_+ (ζ_-) only contains $b_+, b_+^\dagger$ ($b_-, b_-^\dagger$) operators, and different branch creation / annihilation operators commute with each other. For the same reason, any other pair of fermionic field operators from different branches anti-commute amongst each other.
For the $-$ branch, the calculation can be performed in analogy to the $+$-branch calculation.

(b) Substituting the bosonized representation of the fermionic operators into the fermionic form of the free Hamiltonian for right movers with the linear dispersion,

$$\hat{H}_{0,+} = v_F \int dx\, \psi_+^\dagger(x)\left(-\mathrm{i}\frac{\partial}{\partial x} - k_\mathrm{F}\right)\psi_+(x),$$

derive the bosonized form of $\hat{H}_{0,+}$. For this purpose, use the "point-splitting" procedure by first replacing $\psi_+^\dagger(x)$ with $\psi_+^\dagger(x+d)$, expanding the fields in small d, and taking the limit $d \to 0$ at the end of the calculation, with the ground-state average removed as in the normal-ordered product.

Solution: We start with the term

$$\psi_+^\dagger(x)\psi_+(x) \to \psi_+^\dagger(x+d)\psi_+(x). \tag{55}$$

We rewrite this term as we did in subtask (a):

$$\psi_+^\dagger(x+d)\psi_+(x) = \frac{1}{L}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}d}\mathrm{e}^{-\mathrm{i}\eta_+^\dagger(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+(x+d)}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\mathrm{e}^{\mathrm{i}\eta_+(x)} \tag{56}$$

$$= \frac{1}{L}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}d}\mathrm{e}^{-\mathrm{i}(\eta_+^\dagger(x+d)-\eta_+^\dagger(x))}\mathrm{e}^{-\mathrm{i}(\eta_+(x+d)-\eta_+(x))}\mathrm{e}^{[\eta_+(x+d),\eta_+^\dagger(x)]} \tag{57}$$

$$\overset{\lambda\to 0}{\to} \frac{\mathrm{i}}{2\pi d}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}d}\mathrm{e}^{-\mathrm{i}(\eta_+^\dagger(x+d)-\eta_+^\dagger(x))}\mathrm{e}^{-\mathrm{i}(\eta_+(x+d)-\eta_+(x))} \tag{58}$$

We expand in small d:

$$\psi_+^\dagger(x+d)\psi_+(x) \approx \frac{\mathrm{i}}{2\pi d}(1-\mathrm{i}k_\mathrm{F}d)(1-\mathrm{i}\cdot d\cdot\partial_x\eta_+^\dagger(x))(1-\mathrm{i}\cdot d\cdot\partial_x\eta_+(x)) \tag{59}$$

$$\approx \frac{\mathrm{i}}{2\pi d}(1-\mathrm{i}\cdot d\cdot(k_\mathrm{F}+\partial_x\zeta_+(x)) \tag{60}$$

$$= \frac{\mathrm{i}}{2\pi d}+\frac{1}{2\pi}(k_\mathrm{F}+\partial_x\zeta_+(x)) \tag{61}$$

The second term:

$$\psi_+^\dagger(x+d)\partial_x\psi_+(x) = \frac{1}{L}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+^\dagger(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+(x+d)}\left[\partial_x\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\mathrm{e}^{\mathrm{i}\eta_+(x)}\right] \tag{62}$$

$$= \frac{1}{L}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+^\dagger(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+(x+d)}\times \tag{63}$$

$$\times\left[\left[\partial_x\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\right]\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\mathrm{e}^{\mathrm{i}\eta_+(x)}+\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\left[\partial_x\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\right]\mathrm{e}^{\mathrm{i}\eta_+(x)}+\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\left[\partial_x\mathrm{e}^{\mathrm{i}\eta_+(x)}\right]\right] \tag{64}$$

$$= \frac{\mathrm{i}}{L}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+^\dagger(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+(x+d)}\times \tag{65}$$

$$\times\left[k_\mathrm{F}\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\mathrm{e}^{\mathrm{i}\eta_+(x)}+\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}(\partial_x\eta_+^\dagger(x))\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\mathrm{e}^{\mathrm{i}\eta_+(x)}+\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}(\partial_x\eta_+(x))\mathrm{e}^{\mathrm{i}\eta_+(x)}\right] \tag{66}$$

Here we used $[\partial_x\eta_+(x),\eta_+(x)] = [\partial_x\eta_+^\dagger(x),\eta_+^\dagger(x)] = 0$ as can be seen from the definition of $\eta_+(x)$. In this case, the derivative of the exponential can be performed as usual.

$$\psi_+^\dagger(x+d)\partial_x\psi_+(x) = \frac{\mathrm{i}}{L}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}d}\mathrm{e}^{-\mathrm{i}\eta_+^\dagger(x+d)}\mathrm{e}^{-\mathrm{i}\eta_+(x+d)}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\left[k_\mathrm{F}+\partial_x\eta_+^\dagger(x)+(\partial_x\eta_+(x))\right]\mathrm{e}^{\mathrm{i}\eta_+(x)} \tag{67}$$

$$= -\frac{1}{2\pi d}\mathrm{e}^{-\mathrm{i}k_\mathrm{F}d}\mathrm{e}^{-\mathrm{i}\eta_+^\dagger(x+d)}\mathrm{e}^{\mathrm{i}\eta_+^\dagger(x)}\mathrm{e}^{-\mathrm{i}\eta_+(x+d)}\left[k_\mathrm{F}+\partial_x\eta_+^\dagger(x)+(\partial_x\eta_+(x))\right]\mathrm{e}^{\mathrm{i}\eta_+(x)} \tag{68}$$

Now we use the identity

$$[\hat{A},\mathrm{e}^{\hat{B}}] = [\hat{A},\hat{B}]\mathrm{e}^{\hat{B}} \quad \text{for } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0 \tag{69}$$

$$\Rightarrow \mathrm{e}^{\hat{B}}\hat{A} = (\hat{A}-[\hat{A},\hat{B}])\mathrm{e}^{\hat{B}} \tag{70}$$

to get

$$e^{-i\eta_+(x+d)}\partial_x\eta_+^\dagger(x) = (\partial_x\eta_+^\dagger(x) + i[\partial_x\eta_+^\dagger(x), \eta_+(x+d)])e^{-i\eta_+(x+d)} \tag{71}$$

$$[\partial_x\eta_+^\dagger(x), \eta_+(x')] = \partial_x \log\left(\frac{2\pi}{L}[\lambda - i(x'-x)]\right) \tag{72}$$

$$= -i\frac{1}{\lambda - i(x'-x)} \overset{\lambda\to 0}{\to} \frac{1}{x-x'} \tag{73}$$

$$\Rightarrow e^{-i\eta_+(x+d)}\partial_x\eta_+^\dagger(x) = \left(\partial_x\eta_+^\dagger(x) - \frac{i}{d}\right)e^{-i\eta_+(x+d)} \tag{74}$$

thus

$$\psi_+^\dagger(x+d)\partial_x\psi_+(x) = \tag{75}$$

$$= -\frac{1}{2\pi d}e^{-ik_F d}e^{-i(\eta_+^\dagger(x+d)-\eta_+^\dagger(x))}\left[k_F + \partial_x\eta_+^\dagger(x) + \partial_x\eta_+(x) - \frac{i}{d}\right]e^{-i(\eta_+(x+d)-\eta_+(x))} \tag{76}$$

As there is a term of $\mathcal{O}(d^{-2})$ we have to expand the exponentials to second order.

$$-i(\eta_+(x+d) - \eta_+(x)) = -id\left[\partial_x\eta_+(x) + \frac{d}{2}\partial_x^2\eta_+(x)\right] \tag{77}$$

so

$$\psi_+^\dagger(x+d)\partial_x\psi_+(x) \approx -\frac{1}{2\pi d}\left(1-\mathrm{i}k_\mathrm{F}d-\frac{k_\mathrm{F}^2d^2}{2}\right)\left(1-\mathrm{i}d\partial_x\eta_+^\dagger(x)-\frac{d^2}{2}\left(\mathrm{i}\partial_x^2\eta_+^\dagger(x)+(\partial_x\eta_+^\dagger(x))^2\right)\right)\times \tag{78}$$

$$\times\left[k_\mathrm{F}+\partial_x\zeta_+(x)-\frac{\mathrm{i}}{d}\right]\left(1-\mathrm{i}d\partial_x\eta_+(x)-\frac{d^2}{2}\left(\mathrm{i}\partial_x^2\eta_+(x)+(\partial_x\eta_+(x))^2\right)\right) \tag{79}$$

$$\approx\frac{\mathrm{i}}{2\pi d^2}-\frac{1}{2\pi d}\left[(k_\mathrm{F}+\partial_x\zeta_+(x))-\mathrm{i}\left(-\mathrm{i}k_\mathrm{F}-\mathrm{i}\partial_x\zeta_+(x)\right)\right] \tag{80}$$

$$-\frac{1}{2\pi}\left[1112+1123+1211+1222+1321+2111+2122+2221+3121\right] \tag{81}$$

$$=\frac{\mathrm{i}}{2\pi d^2} \tag{82}$$

$$+\frac{\mathrm{i}}{2\pi}\Big[(k_\mathrm{F}+\partial_x\zeta_+(x))\partial_x\eta_+(x)-\frac{1}{2}(\mathrm{i}\partial_x^2\eta_+(x)+(\partial_x\eta_+(x))^2)+\partial_x\eta_+^\dagger(x)(k_\mathrm{F}+\partial_x\zeta_+(x)) \tag{83}$$

$$-\partial_x\eta_+^\dagger(x)\partial_x\eta_+(x)-\frac{1}{2}(\mathrm{i}\partial_x^2\eta_+^\dagger(x)+(\partial_x\eta_+^\dagger(x))^2)+k_\mathrm{F}(k_\mathrm{F}+\partial_x\zeta_+(x)) \tag{84}$$

$$-k_\mathrm{F}\partial_x\eta_+(x)-k_\mathrm{F}\partial_x\eta_+^\dagger(x)-\frac{1}{2}k_\mathrm{F}^2\Big] \tag{85}$$

$$=\frac{\mathrm{i}}{2\pi d^2} \tag{86}$$

$$+\frac{\mathrm{i}}{2\pi}\Big[\frac{k_\mathrm{F}^2}{2}+k_\mathrm{F}\partial_x\zeta_+(x)+\partial_x\zeta_+(x)\partial_x\eta_+(x)-\frac{1}{2}(\partial_x\eta_+(x))^2-\frac{1}{2}(\partial_x\eta_+^\dagger(x))^2-\frac{\mathrm{i}}{2}\partial_x^2\zeta_+(x) \tag{87}$$

$$+\partial_x\eta_+^\dagger(x)\partial_x\zeta_+(x)-\partial_x\eta_+^\dagger(x)\partial_x\eta_+(x)\Big] \tag{88}$$

$$=\frac{\mathrm{i}}{2\pi d^2} \tag{89}$$

$$+\frac{\mathrm{i}}{2\pi}\left[\frac{k_\mathrm{F}^2}{2}+k_\mathrm{F}\partial_x\zeta_+(x)+\partial_x\eta_+^\dagger(x)\partial_x\eta_+(x)+\frac{1}{2}(\partial_x\eta_+(x))^2+\frac{1}{2}(\partial_x\eta_+^\dagger(x))^2-\frac{\mathrm{i}}{2}\partial_x^2\zeta_+(x)\right] \tag{90}$$

$$=\frac{\mathrm{i}}{2\pi d^2}+\frac{\mathrm{i}}{4\pi}[k_\mathrm{F}+\partial_x\zeta_+(x)]^2+\frac{\mathrm{i}}{4\pi}\left[\partial_x\eta_+^\dagger(x)\partial_x\eta_+(x)-\partial_x\eta_+(x)\partial_x\eta_+^\dagger(x)\right]+\frac{1}{4\pi}\partial_x^2\zeta_+(x) \tag{91}$$

evaluating the commutator:

$$\left[\partial_x\eta_+^\dagger(x)\partial_x\eta_+(x)-\partial_x\eta_+(x)\partial_x\eta_+^\dagger(x)\right]=\lim_{x\to x'}\partial_{x'}\left[\partial_x\eta_+^\dagger(x),\eta_+(x')\right] \tag{92}$$

$$=\lim_{x\to x'}\partial_{x'}\frac{-\mathrm{i}}{\lambda-\mathrm{i}(x'-x)} \tag{93}$$

$$=\frac{1}{\lambda^2}. \tag{94}$$

The term $\partial_x^2\zeta_+(x)$ is a total derivative and thus vanishes upon integration (by applying the divergence theorem, and assuming that all fields vanish at infinity).

As instructed, we collect all non-divergent terms and arrive at

$$\hat{H}_{0,+} = v_{\mathrm{F}} \int \mathrm{d}x \, \psi_+^\dagger(x) \left(-\mathrm{i}\frac{\partial}{\partial x} - k_{\mathrm{F}}\right) \psi_+(x) \tag{95}$$

$$= v_{\mathrm{F}} \int \mathrm{d}x \left[-\mathrm{i} : \psi_+^\dagger(x)\partial_x\psi_+(x) : -k_{\mathrm{F}} : \psi_+^\dagger(x)\psi_+(x) :\right] \tag{96}$$

$$= v_{\mathrm{F}} \int \mathrm{d}x \left[\frac{1}{4\pi}[k_{\mathrm{F}} + \partial_x\zeta_+(x)]^2 - \frac{k_{\mathrm{F}}}{2\pi}[k_{\mathrm{F}} + \partial_x\zeta_+(x)]\right] \tag{97}$$

$$= \frac{v_{\mathrm{F}}}{4\pi} \int \mathrm{d}x \left[(\partial_x\zeta_+(x))^2 + \mathrm{const}\right] \tag{98}$$

$$= \pi v_{\mathrm{F}} \int \mathrm{d}x \, \varrho_+^2(x) + \mathrm{const} \tag{99}$$

Here $: \cdot :$ means, that we neglect divergent terms. Again, we removed total derivatives; and const is a constant background density. We identified

$$\partial_x\zeta_+(x) \overset{\lambda\to 0}{\to} -\frac{2\pi}{L}\sum_{q\neq 0} \mathrm{e}^{-\mathrm{i}qx}\varrho_+(q) \tag{100}$$

$$= -\frac{2\pi}{L}\sum_{q} \mathrm{e}^{-\mathrm{i}qx}\varrho_+(q) \tag{101}$$

$$= -(2\pi)\varrho_+(x). \tag{102}$$

2. Zero-bias anomaly in the Luttinger liquid (20 points)

In Sec. 7.7.5 of the Lecture Notes, the energy dependence of the tunneling density of states in a Luttinger liquid was found: $\nu(\epsilon) \propto |\epsilon|^\gamma$, with exponent $\gamma = (K + K^{-1})/2 - 1$ determined by the Luttinger parameter K. Starting with the expression for the Green's function in the spinless case, express $\mathrm{Im}\, G^{\mathrm{R}}(0,t)$ through $G^>$ and $G^<$, and derive the full expression for $\nu(\epsilon)$ at zero temperature, including the numerical factor in front of the power-law energy dependence. The following identities for the Gamma function $\Gamma(x)$ can be useful: $\Gamma(x) = \int_0^\infty \mathrm{d}s \, \exp(-s)s^{x-1}$ and $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$.

Solution: The tunneling density of states (Eq. (7.148) of the lecture notes):

$$\nu_+(\epsilon) = -\frac{1}{\pi}\mathrm{Im}\, G_+^{\mathrm{R}}(0,\epsilon) = -\frac{1}{\pi}\mathrm{Im}\int \mathrm{d}t \, \mathrm{e}^{\mathrm{i}\epsilon t} G_+^{\mathrm{R}}(x=0,t) \tag{103}$$

where G_+^{R} is the retarded Green's function (Eq. (3.77) of the lecture notes):

$$G^{\mathrm{R}}(x,t) = -\mathrm{i}\theta(t)\langle\{\psi_\alpha(x,t), \psi_\alpha^\dagger(0,0)\}\rangle. \tag{104}$$

We introduce the lesser and greater Green's functions and express the retarded and causal Green's functions in terms of them:

$$G_\alpha^>(x,t) := -\mathrm{i}\langle\psi_\alpha(x,t)\psi_\alpha^\dagger(0,0)\rangle \tag{105}$$

$$G_\alpha^<(x,t) := \mathrm{i}\langle\psi_\alpha^\dagger(0,0)\psi_\alpha(x,t)\rangle \tag{106}$$

$$G_\alpha^{\mathrm{R}}(x,t) = \theta(t)[G_\alpha^>(x,t) - G_\alpha^<(x,t)] \tag{107}$$

$$G_\alpha(x,t) = -\mathrm{i}\langle\mathcal{T}\psi_\alpha(x,t)\psi_\alpha^\dagger(0,0)\rangle = \theta(t)G_\alpha^>(x,t) + \theta(-t)G_\alpha^<(x,t). \tag{108}$$

From the Lecture notes (Eq. 7.144) we know the explicit form of the causal Green's function at zero temperature:

$$G_+(x,t) = \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\frac{\lambda^\gamma}{(x-ut+\mathrm{i}\lambda\mathrm{sign}(t))^{1+\gamma/2}(x+ut-\mathrm{i}\lambda\mathrm{sign}(t))^{\gamma/2}} \tag{109}$$

where

$$\gamma = \frac{K+K^{-1}-2}{2}; \tag{110}$$

K is the Luttinger parameter and $u = v_\mathrm{F}/K$. λ is the usual regularization parameter. From $G_+(x,t)$ we read off lesser and greater Green's functions:

$$G_+^>(x,t) = G_+(x,t>0) = \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\frac{\lambda^\gamma}{(x-ut+\mathrm{i}\lambda)^{1+\gamma/2}(x+ut-\mathrm{i}\lambda)^{\gamma/2}} \tag{111}$$

$$G_+^<(x,t) = G_+(x,t<0) = \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\frac{\lambda^\gamma}{(x-ut-\mathrm{i}\lambda)^{1+\gamma/2}(x+ut+\mathrm{i}\lambda)^{\gamma/2}} \tag{112}$$

Thus, we find

$$G_+^\mathrm{R}(x,t) = \frac{\theta(t)\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\lambda^\gamma}{2\pi}\left[\frac{1}{(x-ut+\mathrm{i}\lambda)^{1+\gamma/2}(x+ut-\mathrm{i}\lambda)^{\gamma/2}} - \frac{1}{(x-ut-\mathrm{i}\lambda)^{1+\gamma/2}(x+ut+\mathrm{i}\lambda)^{\gamma/2}}\right] \tag{113}$$

$$= 2\mathrm{i}\frac{\theta(t)\mathrm{e}^{\mathrm{i}k_\mathrm{F}x}\lambda^\gamma}{2\pi}\mathrm{Im}\left[\frac{1}{(x-ut+\mathrm{i}\lambda)^{1+\gamma/2}(x+ut-\mathrm{i}\lambda)^{\gamma/2}}\right] \tag{114}$$

To use this result in Eq. (103), we set $x=0$:

$$G_+^\mathrm{R}(x=0,t) = 2\mathrm{i}\frac{\theta(t)\lambda^\gamma}{2\pi}\mathrm{Im}\left[\frac{1}{(-ut+\mathrm{i}\lambda)^{1+\gamma/2}(ut-\mathrm{i}\lambda)^{\gamma/2}}\right] \tag{115}$$

$$= \mathrm{i}\frac{\theta(t)\lambda^\gamma}{\pi}\mathrm{Im}\left[\frac{(-\mathrm{i})^\gamma}{(-\mathrm{i})^\gamma}\frac{1}{(-1)^{1+\gamma/2}(ut-\mathrm{i}\lambda)^{1+\gamma/2}(ut-\mathrm{i}\lambda)^{\gamma/2}}\right] \tag{116}$$

$$= -\mathrm{i}\frac{\theta(t)}{\pi}\mathrm{Im}\left[\frac{1}{(-\mathrm{i})^\gamma}\frac{(-\mathrm{i}\lambda)^\gamma}{(-1)^{\gamma/2}(ut-\mathrm{i}\lambda)^{1+\gamma}}\right] \tag{117}$$

$$= -\mathrm{i}\frac{\theta(t)}{\pi}\mathrm{Im}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(ut-\mathrm{i}\lambda)^{1+\gamma}}\right] \tag{118}$$

Plugging into Eq. (103):

$$\nu_+(\epsilon) = \frac{1}{\pi^2}\mathrm{Im}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\epsilon t}\mathrm{i}\cdot\mathrm{Im}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(ut-\mathrm{i}\lambda)^{1+\gamma}}\right] \tag{119}$$

$$= \frac{1}{\pi^2}\int_0^\infty \mathrm{d}t\cos(\epsilon t)\mathrm{Im}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(ut-\mathrm{i}\lambda)^{1+\gamma}}\right] \tag{120}$$

$$= \frac{1}{4\mathrm{i}\pi^2}\int_0^\infty \mathrm{d}t\left(\mathrm{e}^{\mathrm{i}\epsilon t}+\mathrm{e}^{-\mathrm{i}\epsilon t}\right)\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(ut-\mathrm{i}\lambda)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(ut+\mathrm{i}\lambda)^{1+\gamma}}\right] \tag{121}$$

$$= \frac{1}{4\mathrm{i}\pi^2 u^{1+\gamma}}\left[\int_0^\infty \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\epsilon t}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(t-\mathrm{i}\lambda/u)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(t+\mathrm{i}\lambda/u)^{1+\gamma}}\right]+\right. \tag{122}$$

$$\left. + \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\epsilon t}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(t-\mathrm{i}\lambda/u)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(t+\mathrm{i}\lambda/u)^{1+\gamma}}\right]\right] \tag{123}$$

$$= \frac{1}{4\mathrm{i}\pi^2 u^{1+\gamma}}\left[\int_0^\infty \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\epsilon t}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(t-\mathrm{i}\lambda/u)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(t+\mathrm{i}\lambda/u)^{1+\gamma}}\right]+\right. \tag{124}$$

$$\left. + \int_{-\infty}^0 \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\epsilon t}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(-t-\mathrm{i}\lambda/u)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(-t+\mathrm{i}\lambda/u)^{1+\gamma}}\right]\right] \tag{125}$$

In the last line we substituted $t \to -t$ in the second term. Rewriting the sum of fractions in the second term:

$$\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(-t-\mathrm{i}\lambda/u)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(-t+\mathrm{i}\lambda/u)^{1+\gamma}}\right] = \left[-\frac{(-\mathrm{i}\lambda)^\gamma}{(-1)^\gamma(t+\mathrm{i}\lambda/u)^{1+\gamma}} + \frac{(\mathrm{i}\lambda)^\gamma}{(-1)^\gamma(t-\mathrm{i}\lambda/u)^{1+\gamma}}\right] \tag{126}$$

$$= \left[\frac{(-\mathrm{i}\lambda)^\gamma}{(t-\mathrm{i}\lambda/u)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(t+\mathrm{i}\lambda/u)^{1+\gamma}}\right] \tag{127}$$

this is the same as the sum of fractions in the first term. Therefore, we have

$$\nu_+(\epsilon) = \frac{1}{4\mathrm{i}\pi^2 u^{1+\gamma}}\int_{-\infty}^\infty \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\epsilon t}\left[\frac{(-\mathrm{i}\lambda)^\gamma}{(t-\mathrm{i}\lambda/u)^{1+\gamma}} - \frac{(\mathrm{i}\lambda)^\gamma}{(t+\mathrm{i}\lambda/u)^{1+\gamma}}\right] \tag{128}$$

$$:= c\int_{-\infty}^\infty \mathrm{d}t I(t) \tag{129}$$

There are two singular points in the integrand: the first fraction has a singularity at $t = \mathrm{i}\lambda/u$, the second one has a singularity at $t = -\mathrm{i}\lambda/u$. As we saw on sheet 0, the function $f(x) = x^\alpha$ has a branch cut on the negative real axis. Thus, the first term features a branch cut at $\mathrm{Im}\,(t) = \lambda/u$, $\mathrm{Re}\,(t) < 0$. The second term has a branch cut at $\mathrm{Im}\,(t) = -\lambda/u$, $\mathrm{Re}\,(t) < 0$. Let us consider the resulting integral first for $\epsilon > 0$. We consider the complex contour integral

$$I_1 := \oint_\Gamma \mathrm{d}s I(s) \tag{130}$$

where the contour Γ consists of

- Γ_1: real axis from $-\infty$ to ∞
- Γ_2: semi-circle counter-clockwise in the upper half plane
- Γ_3: the line immediately above the branch cut at $\mathrm{Im}\,(t) = \lambda/u$, from $-\infty$ to 0
- Γ_4: the line immediately below the branch cut at $\mathrm{Im}\,(t) = \lambda/u$, from 0 to $-\infty$.

Contours $\Gamma_2 - \Gamma_3$, $\Gamma_3 - \Gamma_4$, $\Gamma_4 - \Gamma_2$ are connected by infinitesimally small semi-circles, that give vanishing contributions.
Because of $\epsilon > 0$, Γ_2 does not contribute. This means, that

$$\int_{\Gamma_1} \mathrm{d}s\, I(s) = -\int_{\Gamma_3} \mathrm{d}s\, I(s) - \int_{\Gamma_4} \mathrm{d}s\, I(s) \tag{131}$$

$$\Rightarrow \int_{-\infty}^{\infty} \mathrm{d}t\, I(t) = -\int_{-\infty+\mathrm{i}\lambda/u+\mathrm{i}0}^{0+\mathrm{i}\lambda/u+\mathrm{i}0} \mathrm{d}t\, I(t) + \int_{-\infty+\mathrm{i}\lambda/u-\mathrm{i}0}^{0+\mathrm{i}\lambda/u-\mathrm{i}0} \mathrm{d}t\, I(t) \tag{132}$$

$$= (-\mathrm{i}\lambda)^{\gamma} \mathrm{e}^{-\epsilon\lambda/u} \int_{-\infty}^{0} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\epsilon t} \left[\frac{1}{(t-\mathrm{i}0)^{1+\gamma}} - \frac{1}{(t+\mathrm{i}0)^{1+\gamma}} \right] \tag{133}$$

Here we discarded the second term, because it is continuous in the upper half plane, such that it cancels between contours Γ_3 and Γ_4. Using $(x+\mathrm{i}0)^{1+\gamma} = |t|^{1+\gamma}\mathrm{e}^{\mathrm{i}\pi(1+\gamma)} = -|t|^{1+\gamma}\mathrm{e}^{\mathrm{i}\pi\gamma}$We find

$$\int_{-\infty}^{\infty} \mathrm{d}t\, I(t) = -(-\mathrm{i}\lambda)^{\gamma} \mathrm{e}^{-\epsilon\lambda/u} \int_{-\infty}^{0} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\epsilon t} \frac{1}{|t|^{1+\gamma}} \left[\mathrm{e}^{\mathrm{i}\pi\gamma} - \mathrm{e}^{-\mathrm{i}\pi\gamma}\right] \tag{134}$$

$$= -2\mathrm{i}(\mathrm{i}\epsilon)^{\gamma}\Gamma(-\gamma)(-\mathrm{i}\lambda)^{\gamma}\mathrm{e}^{-\epsilon\lambda/u}\sin(\pi\gamma) \tag{135}$$

$$= -2\mathrm{i}(\lambda\epsilon)^{\gamma}\Gamma(-\gamma)\mathrm{e}^{-\epsilon\lambda/u}\sin(\pi\gamma) \tag{136}$$

where we solved the standard integral with Mathematica. We find

$$\nu_+(\epsilon) = -\frac{(\lambda\epsilon)^{\gamma}}{2\pi^2 u^{1+\gamma}}\Gamma(-\gamma)\mathrm{e}^{-\epsilon\lambda/u}\sin(\pi\gamma) \quad \epsilon > 0. \tag{137}$$

The calculation for $\epsilon < 0$ is analogous. In this case, we close the contour in the lower half plane–excluding the corresponding branch cut—and the branch cut in this plane contributes. In total, we find after taking $\lambda \to 0$ and using $\sin(\pi\gamma)\Gamma(-\gamma) = -\frac{\pi}{\Gamma(1+\gamma)}$

$$\nu_+(\epsilon) = \frac{1}{2\pi u\Gamma(1+\gamma)}\left(\frac{\lambda|\epsilon|}{u}\right)^{\gamma}. \tag{138}$$

We read off the prefactor from the lecture notes (7.149): $[2\pi\Gamma(1+\gamma)]^{-1}$.