Karlsruher Institut für Technologie   Institute for Condensed Matter Theory
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Condensed Matter Theory II: Many-Body Theory (TKM II)   SoSe 2023

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Homework assignment 13
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1. Conductance of a non-interacting quantum wire (10 + 10 = 20 point)

In Sec. 7.8.1 of the Lecture Notes, an impurity in a quantum wire was described by the following term in the Hamiltonian: $H_{\text{imp}} = \int dx\, \mathcal{U}(x)\psi^\dagger(x)\psi(x)$, where $\psi(x) = \psi_+(x) + \psi_-(x)$ is the total fermionic operator, including right-moving and left-moving contributions. The impurity was then characterized by the forward and backward scattering amplitudes

$$\mathcal{U}_{\text{f}} = \mathcal{U}(k=0) \equiv \int dx\, \mathcal{U}(x), \tag{1}$$

$$\mathcal{U}_{\text{b}} = \mathcal{U}(k=2k_F) \equiv \int dx\, \mathcal{U}(x) e^{-2ik_F x}. \tag{2}$$

Consider a quantum wire connected to two reservoirs biased by the voltage V. The interfaces between the wire and the reservoirs do not lead to the electron backscattering. The conductance g of the wire is given by the ratio of the electrical current $I = \frac{e}{h}\int \mathrm{d}k \frac{\partial \varepsilon_k}{\partial k} n_k$ and the voltage V: $g = I/V$. Calculate the conductance of a non-interacting wire at zero temperature...

(a) in the absence of the impurity potential (clean wire).

Solution: The current is given by

$$I = -\frac{e}{h}\int_k v_k n_k \tag{3}$$

$$= -\frac{e}{h}\int_{k>0} v_k (n_k - n_{-k}) \tag{4}$$

In 1D, the group velocity $v_k = \partial\varepsilon_k/\partial k$ and density of states $\nu(\epsilon) = (\partial\varepsilon_k/\partial k)^{-1}$ exactly cancel against each other. The assumption about no backscattering means that we assume that the occupation number n_k for right-movers is equal to the occupation number in the left reservoir, and the occupation number n_{-k} for left-movers equals that of the right reservoir. For the current we get

$$I = -\frac{e}{h}\int \mathrm{d}\varepsilon \left[\theta(\varepsilon - \mu_L) - \theta(\varepsilon - \mu_R)\right] \tag{5}$$

$$= -\frac{e}{h}\underbrace{(\mu_L - \mu_R)}_{-eV} \tag{6}$$

$$= \frac{e^2}{h}V \tag{7}$$

Where we have identified the voltage V. We find the conductance

$$g = \frac{I}{V} = \frac{e^2}{h}, \tag{8}$$

which is half the conductance quantum, a quantity generally relevant for ballistic conductors. The usual factor of 2 would comes from spin, which we do not have here.

(b) in the presence of a single (weak) impurity in the wire.

Solution: According to lecture Eq. (7.165) the impurity is represented by a δ-potential, with amplitude $\mathcal{U}_\mathrm{b}$ for the backscattering part. Forward scattering can be neglected.

Our 1D model was originally derived from a model with a parabolic spectrum, and we will use a dispersion $E = p^2/2m$ here also, in order to use the usual results for the potential scattering. Besides, it is not immediately clear how to write the Schrödinger equation in position space for Tomonaga or Luttinger models.

Scattering on a δ-barrier was solved in QM I: Assuming an incoming wave $\psi_+(k)$ from the left of the barrier, there is a reflected part $r_k\psi_+(k)$ with

$$r_k = \frac{1}{\frac{\mathrm{i}k}{m\mathcal{U}_\mathrm{b}} - 1}. \tag{9}$$

We assume that the voltage is small enough so that the momentum will be close to the Fermi momentum and approximate

$$\frac{k}{m} \approx v_F. \tag{10}$$

The reflection coefficient is given to the leading order $\mathcal{U}_\mathrm{b}$ by

$$r \approx -\mathrm{i}\frac{\mathcal{U}_\mathrm{b}}{v_F}. \tag{11}$$

Due to current conservation, the current in 1D system is position independent, and we calculate it left of the barrier. As we are considering elastic scattering, we only have to take into account energies that are not fully occupied:

$$I = -\frac{e}{h}\int_{\mu_L}^{\mu_R} \mathrm{d}\varepsilon \, [n_k - n_{-k}] \tag{12}$$

The occupation number for right-movers is not affected on the left side of the barrier, and on this interval, we have $n_k = 1$. Due to reflection, the population of the right-movers is not zero anymore, but is given by

$$n_{-k} = |r_k|^2 n_k = |r|^2 \tag{13}$$

Thus we get

$$g = \frac{I}{V} = -\frac{e}{hV}\int_{\mu_L}^{\mu_R} \mathrm{d}\varepsilon \left(1 - |r|^2\right) \tag{14}$$

$$= \frac{e^2}{h}(1 - |r|^2) \tag{15}$$

$$\approx \frac{e^2}{h}\left(1 - \frac{\mathcal{U}_\mathrm{b}^2}{v_\mathrm{F}^2}\right). \tag{16}$$

2. Interaction-induced backscattering (15 points)

In Sec. 7.9.1 of Lecture Notes, interaction-induced backscattering in a spinful Luttinger liquid was introduced. After bosonization, the interaction Hamiltonian is

$$H_{1\perp} = \frac{2g_{1\perp}}{(2\pi\lambda)^2}\int \mathrm{d}x \cos[2\sqrt{2}\phi_\sigma(x)]. \tag{17}$$

Considering the coupling g as small, derive the RG equation

$$\frac{dg_{1\perp}}{d\ln b} = (2-2K_\sigma)g_{1\perp}\,.$$

in the same way as it was done for the impurity-induced backscattering amplitude $\mathcal{U}_\mathrm{b}$ in Sec. 7.8.1 of Lecture Notes.

Solution:

To simplify notation, we drop all the σ's and subscripts from g. The total action is $S = S_0 + S_{1\perp}$, where

$$S_0[\phi] = \frac{1}{2\pi uK}\int \mathrm{d}\tau\mathrm{d}x[(\partial_\tau\phi)^2 + u^2(\partial_x\phi)^2] \tag{18}$$

To derive the RG equation, we follow the steps of Sec. 7.8.1. Since g is small, we make the expansion

$$e^{-S[\phi]} \approx e^{-S_0[\phi]-S_\mathrm{imp}[\phi]} \approx e^{-S_0[\phi]}(1-S_\mathrm{imp}[\phi]). \tag{19}$$

In our theory, we have some ultraviolet cutoff λ. We label the momenta in the interval $[(b\lambda)^{-1},\lambda^{-1}]$ as *fast*, and the smaller momenta as *slow*. We can sepate the free part of the action into two:

$$S_0[\phi] = S_0[\phi_>] + S_0[\phi_<]. \tag{20}$$

Considering just the integral over the fast fields gives

$$\begin{aligned}\int \mathcal{D}\phi^> e^{-S[\phi^>,\phi^<]} &= e^{-S_0[\phi^<]}\int \mathcal{D}\phi^>\left(1-S_\mathrm{imp}[\phi]\right)e^{-S_0[\phi^>]}\\ &= e^{-S_0[\phi^<]}\left(1-\langle S_\mathrm{imp}[\phi]\rangle_>\right),\end{aligned} \tag{21}$$

where the functional integral is given by

$$\langle S_\mathrm{imp}[\phi]\rangle_> = \frac{2g}{(2\pi\lambda)^2}\int \mathrm{d}\tau\mathrm{d}x\left\langle\cos[2\sqrt{2}\phi(x,\tau)]\right\rangle_> \tag{22}$$

$$= \frac{2g}{(2\pi\lambda)^2}\mathrm{Re}\int \mathrm{d}\tau\mathrm{d}x\left\langle e^{\mathrm{i}\,2\sqrt{2}\phi(x,\tau)}\right\rangle_> \tag{23}$$

$$= \frac{2g}{(2\pi\lambda)^2}\mathrm{Re}\int \mathrm{d}\tau\mathrm{d}x e^{\mathrm{i}\,2\sqrt{2}\phi^<(x,\tau)}\left\langle e^{\mathrm{i}\,2\sqrt{2}\phi^>(x,\tau)}\right\rangle_> \tag{24}$$

$$\tag{25}$$

The above average is simply a gaussian integral in which the argument of the exponential inside the angle bracket acts as a source term. We use Eq. (6.55) to evaluate it. $A(q,\omega) = \frac{q^2+\omega^2}{\pi uK}$ and $J(q) = -\mathrm{i}2\sqrt{2}$

$$\left\langle e^{\mathrm{i}\,2\sqrt{2}\phi^>(x,\tau)}\right\rangle_> = \int \mathcal{D}\phi^> \exp\left[\int_{q,\omega}\left(\frac{u^2q^2+\omega^2}{2\pi uK}[\phi^>(q,\omega)]^2 + \mathrm{i}2\sqrt{2}\phi^>(q,\omega)\right)\right] \tag{26}$$

$$\simeq \exp\left(-4\int_{(b\lambda)^{-1}<q<\lambda^{-1}}\frac{\pi uK}{u^2q^2+\omega^2}\right) \tag{27}$$

The determinant prefactor we have dropped at '$\simeq$' is the same as for the average $\langle 1\rangle_> \simeq 1$ done implicitly in Eq. (21). The above exponential factor determines the relative magnitude of 1 and $S_{\text{imp}}[\phi_<]$

In the following, we assume that the fast fields are defined so that the field is fast if $(b\lambda)^{-1} < u^2q^2+\omega^2$, *i.e.* either momentum or frequency can be large. Then we can reduce the integral to a spherically symmetric 2D integral:

$$\int_{(b\lambda)^{-1}<q<\lambda^{-1}} \frac{\pi u K}{u^2q^2+\omega^2} = \int \frac{\mathrm{d}q\,\mathrm{d}\omega}{(2\pi)^2}\frac{\pi u K}{u^2q^2+\omega^2} = \frac{K}{2}\int_{(b\lambda)^{-1}}^{\lambda^{-1}} \frac{\mathrm{d}r\, r}{r^2} = \frac{K}{2}\ln b \tag{28}$$

Substituting this back into Eq. (27) gives

$$\exp\left(-4\cdot\frac{K}{2}\ln b\right) = b^{-2K}. \tag{29}$$

Now the impurity action becomes

$$\langle S_{\text{imp}}[\phi]\rangle_> = \frac{2gb^{-2K}}{(2\pi\lambda)^2}\int \mathrm{d}\tau\mathrm{d}x\cos[2\sqrt{2}\phi^<(x,\tau)]. \tag{30}$$

Now we rescale

$$x' = x/b, \qquad \tau' = \tau/b \tag{31}$$

$$\phi'(x',\tau') = \phi^<(x,\tau) \tag{32}$$

We get a

$$\langle S_{\text{imp}}[\phi]\rangle_> = \frac{2gb^{2-2K}}{(2\pi\lambda)^2}\int \mathrm{d}\tau'\mathrm{d}x'\cos[2\sqrt{2}\phi'(x',\tau')] \tag{33}$$

which is of the same form as before.

As a function of b, we have

$$g(b) = b^{2-K}g(0), \tag{34}$$

where $g(0)$ is the original coupling constant. Defining a logarithmic variable L as $b = e^L$, we write above as

$$g(L) = e^{(2-2K)L}g(0) \tag{35}$$

Take the derivative around $L = 0$ to obtain the RG equation

$$\frac{\mathrm{d}g}{\mathrm{d}L} = (2-2K)g. \tag{36}$$

This grows infinitely strong if $K < 1$, or vanishes asymptotically if $K > 1$. In the end, there is not much difference to the single impurity case. Now the interaction acts as a scatterer that is present everywhere in space, but the calculation remain the same.

3. Kubo formula for the conductivity (15 points)

In Sec. 8.2 of Lecture Notes, the Kubo formula for the conductivity of non-interacting fermions was derived. Starting from the Matsubara expression for the current-current response function,

$$\mathscr{D}^M_{jj;\mu\nu}(\mathbf{r},\mathbf{r}';\omega_m) = e^2\frac{1}{\beta}\sum_{\varepsilon_n} \hat{v}_\mu\,\mathscr{G}_{M,0}(\mathbf{r},\mathbf{r}';\varepsilon_n+\omega_m)\,\hat{v}_\nu\,\mathscr{G}_{M,0}(\mathbf{r}',\mathbf{r},\varepsilon_n).$$

derive the retarded response function $\mathscr{D}^R_{jj;\mu\nu}(\mathbf{r},\mathbf{r}';\omega)$ [Eq. (8.31) of Lecture Notes].

Solution:

The position coordinates will stay fixed in the following, and we can omit them. We will also omit the operators $\hat{\nu}$ and the prefactor e^2. Let us define the GF

$$G(z) = \mathscr{G}_{M,0}(-iz), \tag{37}$$

so that $G(\mathrm{i}\varepsilon_n) = \mathscr{G}_{M,0}(\varepsilon_n)$. G is an analytic function separately on the upper half plane ($\operatorname{Im} z > 0$, we denote this by G^R) and on the lower half plane ($\operatorname{Im} z < 0$, denoted by G^A). The current-current response function can be written as

$$\begin{aligned}\mathscr{D}^M_{jj;\mu\nu}(\omega_m) &= \frac{1}{\beta}\sum_{\varepsilon_n} G(\mathrm{i}\varepsilon_n + \mathrm{i}\omega_m)G(\mathrm{i}\varepsilon_n)\\ &= \oint_{\mathcal{C}_+}\frac{\mathrm{d}z}{2\pi i} n_F(z)G^R(z)G^R(z+\mathrm{i}\omega_m)\\ &+ \oint_{\mathcal{C}_0}\frac{\mathrm{d}z}{2\pi i} n_F(z)G^A(z)G^R(z+\mathrm{i}\omega_m)\\ &+ \oint_{\mathcal{C}_-}\frac{\mathrm{d}z}{2\pi i} n_F(z)G^A(z)G^A(z+\mathrm{i}\omega_m).\end{aligned} \tag{38}$$

On the last step, we transformed the sum into an contour intergral, similar to Ex.3(e) from Sheet 7. But now the integrand has two lines of non-analyticity, one at $z = 0$ and one at $z = -\mathrm{i}\omega_m$, and we need three contours. The first one, $\mathcal{C}_+$, is a counterclockwise semicircle on the upper half plane connected by a straight line from $z = -\infty + \mathrm{i}0^+$ to $z = +\infty + \mathrm{i}0^+$. The third one, $\mathcal{C}_-$, is a counterclockwise semicircle on the lower half plane connected by a straight line from $z = +\infty - \mathrm{i}(\omega_m + 0^+)$ to $z = -\infty - \mathrm{i}(\omega_m + 0^+)$. The middle one, $\mathcal{C}_0$ includes straight lines from $z = -\infty - \mathrm{i}(\omega_m - 0^+)$ to $z = +\infty - \mathrm{i}(\omega_m - 0^+)$ and from $z = +\infty - \mathrm{i}0^+$ to $z = -\infty - \mathrm{i}0^+$, connected by segments at infinity.

The semicircles and the segments are assumed to vanish, and we are left with four integrals over the real line:

$$\begin{aligned}\mathscr{D}^M_{jj;\mu\nu}(\omega_m) = -\mathrm{i}\Bigg[&\int\frac{\mathrm{d}E}{2\pi} n_F(E)G^R(E)G^R(E+\mathrm{i}\omega_m)\\ &-\int\frac{\mathrm{d}E}{2\pi} n_F(E)G^A(E)G^R(E+\mathrm{i}\omega_m)\\ &+\int\frac{\mathrm{d}E}{2\pi} n_F(E-\mathrm{i}\omega_m)G^A(E-\mathrm{i}\omega_m)G^R(E)\\ &-\int\frac{\mathrm{d}E}{2\pi} n_F(E-\mathrm{i}\omega_m)G^A(E-\mathrm{i}\omega_m)G^A(E)\Bigg].\end{aligned} \tag{39}$$

As usual, the bosonic Matsubara frequency ω_m can be removed from the Fermi function. We may then do the analytical continuation from $\mathrm{i}\omega_m \to -\omega$ to obtain

$$\begin{aligned}D(\omega) = \mathscr{D}^M_{jj;\mu\nu}(\mathrm{i}\omega) = -\mathrm{i}\int\frac{\mathrm{d}E}{2\pi} n_F(E)\Big[&G^R(E)G^R(E-\omega) - G^A(E)G^R(E-\omega)\\ &+G^A(E+\omega)G^R(E) - G^A(E+\omega)G^A(E)\Big].\end{aligned} \tag{40}$$

which can be regrouped to give

$$
\begin{aligned}
D(\omega) = -\mathrm{i} \int \frac{\mathrm{d}E}{2\pi} n_F(E) \Big\{ &[G^R(E) - G^A(E)] G^R(E+\omega) \\
&+ G^A(E+\omega)[G^R(E) - G^A(E)] \Big\} \\
= -\mathrm{i} \int \frac{\mathrm{d}E}{2\pi} \Big\{ & f(E+\omega)[G^R(E+\omega) - G^A(E+\omega)] G^R(E) \\
&+ G^A(E+\omega)[G^R(E) - G^A(E)] f(E) \Big\}.
\end{aligned}
\tag{41}
$$

This is, up to the factors that we have not written explicitly, Eq. (8.31) of the lecture notes.

There is some structure here. The combination

$$
G^K(E) = [G^R(E) - G^A(E)] f(E) \tag{42}
$$

is the Keldysh part of the Green's function for a system in equilibrium. Using this, the Green's function can be written compactly as

$$
D(\omega) = -\mathrm{i} \int \frac{\mathrm{d}E}{2\pi} \Big\{ G^K(E+\omega) G^R(E) + G^A(E+\omega) G^K(E) \Big\}. \tag{43}
$$

One could have obtained this result also via another route using the heavy machinery of the non-equilibrium field theory (more specifically Keldysh technique). But analytically continuing Matsubara Green's functions is arguably a simpler way.