Einführung in Theoretische Teilchenphysik

Lectures: Prof. Dr. M. M. Mühlleitner – Exercises: M.Sc. Martin Gabelmann, Dr. Sophie Williamson

Exercise Sheet 2

Hand-in Deadline: Friday 20.11.20, 14:00.
Discussion: Tuesday 24.11.20, Thursday 26.11.20.

1. **[7 points] Tensors under Lorentz transformations**: Taking $x \to {x'}^\mu = \Lambda^\mu{}_\nu x^\nu$ and the orthogonality condition, $\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma g_{\mu\rho} = g_{\nu\sigma}$,

   (a) **[1 point]** Show that $a^\mu b_\mu$ is invariant under Lorentz transformations.

   (b) **[2 points]** Let $(\Lambda^{-1})^\mu{}_\nu$ be the inverse of $\Lambda_\nu{}^\mu$, meaning that

   $$(\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\rho = \delta^\mu_\rho \,.$$

   Show that $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$.

   (c) **[2 points]** Show that ${\partial_\mu}' = (\Lambda^{-1})^\nu{}_\mu \partial_\nu$ where $\partial'_\mu = \frac{\partial}{\partial {x'}^\mu}$.

   (d) **[2 points]** Show that for a general 2-index tensor $M_{\mu\nu}$, that $M^\mu{}_\nu$ and $M_\nu{}^\mu$ are not necessarily equal. On the other other hand, show that $\delta^\mu{}_\nu = \delta_\nu{}^\mu$. Why is the ordering not relevant in the latter case?

2. **[13 points] Coupled Quantum Harmonic Oscillators** Take a 1-dimensional quantum mechanical chain of $n = 0...N$ particles, each of which is in a harmonic potential, with mass m and a distance a to the next particle when in the equilibrium position. Each set of neighbouring particles is coupled to each other. The displacement (in the x-direction) of each particle from its equilibrium position is given by q_n, and has an associated momenta p_n. The chain is closed, such that $q_0 = q_N$. In natural units, the Hamilton operator for such a particle chain is

   $$H = \sum_{n=1}^{N} \frac{1}{2m} p_n^2 + \frac{m\Omega^2}{2}(q_n - q_{n-1})^2 + \frac{m\Omega_0^2}{2} q_n^2 \,,$$

   where Ω is the strength of the coupling between the nearest neighbours.

   (a) **[1 point]** The x-coordinates of the n^{th} mass point is given by $x_n = a_n + q_n = n\,a + q_n$ with $[x_n, p_m] = i\delta_{nm}$. What are the following commutators?

   $$[q_n, p_m], \qquad [q_n, q_m], \qquad [p_n, p_m] \,.$$

   (b) **[3 points]** From the Hamilton operator, derive the equations of motion associated with the system in the Heisenberg representation (time-dependent operators). Combine them into a common differential equation of second-order.

Hint: Recall that in the Heisenberg picture of quantum mechanics, the state vectors do not change with time, while the observables absorb the time dependence:

$$\frac{d}{dt}A(t) = i[H, A(t)] + \left(\frac{\partial}{\partial t}A\right)_H , \tag{1}$$

with the operator $A(t)$ evolving according to $A(t) = U^\dagger(t)AU(t)$. Start by writing the particle displacements and momenta in this form.

(c) **[3 points]** For the diagonalisation of the Hamilton operator, we introduce so-called 'normal' coordinates Q_k and impulse P_k,

$$\begin{aligned} q_n &= \frac{1}{\sqrt{mN}}\sum_k e^{ika_n}Q_k , \qquad &\to Q_k = \sqrt{\frac{m}{N}}\sum_n e^{-ika_n}q_n , \\ p_n &= \sqrt{\frac{m}{N}}\sum_k e^{-ika_n}P_k , \qquad &\to P_k = \frac{1}{\sqrt{mN}}\sum_n e^{ika_n}p_n . \end{aligned}$$

as Fourier sums.

i. Reason that, due to periodic boundary conditions $e^{ikaN} = 1$, the summation index, $k = \frac{2\pi l}{Na}$, is constrained by $l = (-\frac{N}{2}, +\frac{N}{2}]$.

ii. Using the boundary conditions, show that the Fourier coefficients have the completeness and orthogonality relations,

$$\frac{1}{N}\sum_{n=1}^{N} e^{ikan}e^{-ik'an} = \delta_{kk'} , \qquad \frac{1}{N}\sum_k e^{ikan}e^{-ikan'} = \delta_{nn'} .$$

(d) **[3 points]** In terms of normal coordinates and the impulse, the Hamilton operator can be written like

$$H = \frac{1}{2}\sum_k \left(P_k P_k^\dagger + \omega_k^2 Q_k Q_k^\dagger\right) ,$$

where ω_k is the angular frequency of the k^{th} oscillator. Due to the hermicity of q_n and p_n, $Q_k^\dagger = Q_{-k}$ and $P_k^\dagger = P_{-k}$.
Introducing creation and annihilation operators,

$$a_k = \frac{1}{\sqrt{2\omega_k}}\left(\omega_k Q_k + iP_k^\dagger\right) , \qquad a_k^\dagger = \frac{1}{2\omega_k}\left(\omega_k Q_k^\dagger - iP_k\right) ,$$

Express the Hamilton operator in terms of these.
Hint: First find Q_k and P_k as functions of a_k, $a_{-k}^\dagger$, a_{-k}, $a_k^\dagger$, then calculate the commutators $[a, a_{k'}]$, $[a_k^\dagger, a_{k'}^\dagger]$, $[a_k^\dagger, a_{k'}]$ before inserting the expressions back into that for the Hamilton operator.

(e) **[2 points]** Consider the continuum limit in which $a \to 0$ and $N \to \infty$, where the discretised steps between the coupled oscillators tend towards the continuous spectrum of a vibrating string with length $L = aN$, density $\rho = \frac{m}{a}$ and tension $v^2 = (\Omega a)^2$ – each of which stay constant. Rewrite the equations of motion for the string with

$$q(x) = q_n\sqrt{\frac{m}{a}} , \qquad p(x) = p_n\sqrt{\frac{1}{ma}} .$$

Hint: First consider the meaning of $q_{n\pm 1}$.

(f) **[1 point]** Consider the generalisation of the 1-dimensional system to 3 (spatial)-dimensions, $x \to \mathbf{x}$, with the oscillator displacements remaining solely 1-dimensional, by making the substitutions

$$v \to c , \qquad \frac{\Omega_0^2}{c^2} \to m^2 , \qquad (\mathbf{x}, t) \equiv x , \qquad q(\mathbf{x}, t) \to \phi(x) .$$

Rewrite the equations of motion. Which equation do you find?