Einführung in Theoretische Teilchenphysik

Lectures: Prof. Dr. M. M. Mühlleitner – Exercises: M.Sc. Martin Gabelmann, Dr. Sophie Williamson

Exercise Sheet 3

Hand-in Deadline: Friday 27.11.20, 14:00.
Discussion: Tuesday 1.12.20, Thursday 3.12.20.

1. [**4 points**] **Invariance of the scalar field Lagrangian**: Consider the transformation, $\mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu(\phi)$, of the Lagrangian density $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$, where Λ^μ are arbitrary functions of the scalar field $\phi(x)$. Show that this transformation leaves the Euler-Lagrange equations of motion unchanged.

2. [**5 points**] **Noether's Theorem**: Noether's theorem states that each continuous symmetry transformation that leaves the action invariant leads to a conservation law and a conserved quantity, such as $\partial_\mu J^\mu(x) = 0$ and $Q = \int d^3x J^0(x)$, respectively.

   (a) [**2 points**] Show that the Lagrangian density,
   $$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 ,$$
   of two real scalar fields, ϕ_1 and ϕ_2, is invariant under the transformation,
   $$\begin{aligned} \phi_1 \to \phi_1' &= \phi_1 \cos\theta + \phi_2 \sin\theta , \\ \phi_2 \to \phi_2' &= -\phi_1 \sin\theta + \phi_2 \cos\theta , \\ x_\mu \to x_\mu' &= x_\mu , \end{aligned}$$
   where θ is a real constant.

   (b) [**2 points**] Calculate the corresponding Noether current, J^μ, and charge, Q.

   (c) [**1 point**] Find an expression for the energy-momentum tensor, $T_{\mu\nu}$, and write down T_{00} and T_{ii}.

3. [**8 points**] **Lagrangian of a massless vector field**: When considering a quantum description of a theory, the freedom generally afforded to choosing co-ordinates should be fixed, so that one representative state is chosen from all possible physically identical states. The quantisation of gauge fields, $A^\mu(x)$, removes any redundant degrees of freedom by imposing a local constraint at each point in space-time. This is the process of *gauge-fixing*. It should be noted that the choice of gauge has no effect on physical observables, which are all gauge-independent. The Lagrangian of a massless gauge field in the presence of a source, J^μ, is given by
   $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{2}(\partial_\mu A^\mu)^2 - J_\mu A^\mu ,$$
   where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ denotes the field-strength tensor, and α is the gauge-fixing parameter – an arbitrary constant.

(a) **[3 points]** Derive the equations of motion for A_μ, and show that the result yields the Maxwell equations plus a gauge condition.

(b) **[2 points]** The Gupta-Bleuler Lagrangian is given by

$$\tilde{\mathcal{L}} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - J_\mu A^\mu \,.$$

Show that $\tilde{\mathcal{L}}$ yields the same field equations as in (a) when $\alpha = 1$.

(c) **[3 points]** Compute $\tilde{\mathcal{L}} - \mathcal{L}(\alpha = 1)$, and state how the result explains why the equations of motion for both Lagrangian densities are the same.

4. **[3 points] Gamma-matrix manipulation**: Using the fact that $\{\gamma^5, \gamma^\mu\} = 0$, and $\mathrm{tr}(\mathbb{1}_4) = 4$, show that

(a) $\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma^5) = 0\,,$

(b) $\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}\,,$

(c) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0\,,$

(d) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4g^{\mu\nu} g^{\rho\sigma} + 4g^{\mu\sigma} g^{\nu\rho} - 4g^{\mu\rho} g^{\nu\sigma}\,.$