Karlsruher Institut für Technologie

Einführung in Theoretische Teilchenphysik

Lectures: Prof. Dr. M. M. Mühlleitner – Exercises: M.Sc. Martin Gabelmann, Dr. Sophie Williamson

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Exercise Sheet 4

Hand-in Deadline: Friday 4.12.20, 14:00.
Discussion: Tuesday 8.12.20, Thursday 10.12.20.

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1. **[13 points] Canonical quantisation of a real scalar field**:
A real scalar field is governed by the Klein-Gordon Lagrangian

$$\mathcal{L}_{\rm KG} = \frac{1}{2}(\partial_\mu \phi(x))(\partial^\mu \phi(x)) - \frac{m^2}{2}\phi(x)^2 \,.$$

The Fourier transformation of the quantised field, $\phi(x)$, can be written as

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left[a(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{+ipx}\right] \,,$$

where the plane wave solutions obey the orthogonality condition:

$$\int d^3x e^{-ipx}e^{ip'x} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}\,') \,.$$

(notice that, due to the energy-momentum relation $E_p = \sqrt{\vec{p}^{\,2}+m^2}$, the above condition implies $p_0' = p_0$).

(a) **[2 points]** Show that the annihilation operator takes the form

$$a(\vec{p}) = \int d^3x\, e^{ipx}\left(i\Pi(x) + E_p\phi(x)\right) \,,$$

where $\Pi \equiv \partial\mathcal{L}/\partial(\partial_0\phi)$ is the canonical momentum.

(b) **[4 points]** Given in terms of the fields, the canonical commutation relations yield

$$[\phi(\vec{x},t),\phi(\vec{x}\,',t)] = [\Pi(\vec{x},t),\Pi(\vec{x}\,',t)] = 0 \,, \qquad [\phi(\vec{x},t),\Pi(\vec{x}\,',t)] = i\delta^{(3)}(\vec{x}-\vec{x}\,') \,.$$

Obtain the corresponding relations for the operators $a(\vec{p}), a^\dagger(\vec{p})$,

$$[a(\vec{p}),a(\vec{p}\,')] = [a^\dagger(\vec{p}),a^\dagger(\vec{p}\,')] = 0 \,, \qquad [a(\vec{p}),a^\dagger(\vec{p}\,')] = (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{p}\,') \,.$$

(c) **[1 point]** Write down an expression for the Stress-Energy-Momentum tensor $T^\mu{}_\nu$ of the scalar field $\phi(\vec{x},t)$.

(d) **[4 points]** The energy spectrum of the scalar field is given by the eigenvalues of the Hamilton operator $\hat{H} = \int d^3x \mathcal{H} = \int d^3x T^0_0$. Using the Fourier transformation of the field, along with the commutation relations for $a(\vec{p}), a^\dagger(\vec{p})$, prove that $\hat{H}$ is diagonalised and may be written as

$$\hat{H} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p \left[2\tilde{N}(\vec{p}) + C \right] ,$$

where here we have introduced the particle number operator,

$$N = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} a^\dagger(\vec{p}) a(\vec{p}) = \int \frac{d^3p}{(2\pi)^3} \tilde{N}(\vec{p}) ,$$

with C standing for a constant term. What is its physical interpretation?

(e) **[2 points]** Similarly, show that the 3-momentum of the field, $P_i = \int d^3x T^0_i$, yields

$$\vec{P} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \vec{p} \left[2\tilde{N}(\vec{p}) + C \right] .$$

2. **[7 points] Tensor formulation of classical electrodynamics**:

(a) **[2 points]** Given explicitly the electromagnetic tensor,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} , \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} ,$$

calculate the dual, $\tilde{F}^{\mu\nu}$, where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor,

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \epsilon^{0123}, \dots \\ -1 & \epsilon^{0132}, \dots \\ 0 & \epsilon^{0012}, \dots \end{cases}$$

(b) **[4 points]** Given that $\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$ and $\partial_\mu F^{\mu\nu} = J^\nu$, show how one can obtain the Maxwell's equations (in Heavyside-Lorentz units $c = \epsilon_0 = \mu_0 = 1$),

$$\vec{\nabla} \cdot \vec{E} = \rho , \qquad \vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 , \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 .$$

Hint: You may find it useful to recall the Bianchi identity,

$$\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0 .$$

(c) **[1 point]** Show that $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \propto F_{\mu\nu}F^{\mu\nu}$ and determine the proportionality constant.