Einführung in Theoretische Teilchenphysik

Lectures: Prof. Dr. M. M. Mühlleitner – Exercises: M.Sc. Martin Gabelmann, Dr. Sophie Williamson

Exercise Sheet 5

Hand-in Deadline: Friday 11.12.20, 14:00.
Discussion: Tuesday 15.12.20, Thursday 17.12.20.

1. **[12 points] Spinor Algebra**: Consider the spinors

$$u(\vec{p}, s) = \sqrt{p_0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi_s \end{pmatrix}, \qquad \chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$v(\vec{p}, s) = \sqrt{p_0 + m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi_s \\ \chi_s \end{pmatrix}, \qquad \chi_{\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where s is the spin of the of the field, $s \in \{-\frac{1}{2}, \frac{1}{2}\}$, $p_0 = \sqrt{m^2 + \vec{p}^2}$ the energy, and $\vec{\sigma}$ denotes the Pauli matrices,

$$\vec{\sigma} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right].$$

(a) **[2 points]** To calculate experimental observables such as the cross section, we need to fix the normalisation of the free particle wave-function. For covariant normalisation, we choose to normalise to $2p_0$ particles per unit volume,

$$\int_V \rho \, dV = 2\, p_0 \,.$$

Taking the equation for $u(\vec{p}, s)$ and replacing the prefactor $\sqrt{p_0 + m}$ with N, show that in the scheme of covariant normalisation, the normalisation constant N for the Dirac spinor is indeed $\sqrt{p_0 + m}$.
Hint: Start with a plane-wave solution for a free particle, $\psi(\vec{p}) = u(\vec{p})e^{-ipx}$.

(b) **[1 point]** Prove that

$$(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) = \vec{p}^{\,2} \mathbb{1}_{2\times 2}$$

(c) **[1 point]** The Dirac matrices in the Dirac representation, γ^μ_D ($\mu = 0...3$), are given by

$$\gamma^\mu_D = \left(\begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right).$$

Show explicitly that the anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ holds for the representation.

(d) **[2 points]** Furthermore, prove using the explicit Dirac representation that the spinors $u(\vec{p}, s)$ and $v(\vec{p}, s)$ are solutions to the Dirac equations,

$$\begin{aligned} (\not{p} - m)u(\vec{p}) &= 0 \\ (\not{p} + m)v(\vec{p}) &= 0\,. \end{aligned}$$

(e) **[3 points]** Prove the orthogonality relations for the spinors $u(\vec{p}, s)$ and $v(\vec{p}, s)$:

$$\begin{aligned} u^\dagger(\vec{p}, s)u(\vec{p}, s') = 2p_0\delta_{ss'}\,, &\qquad v^\dagger(\vec{p}, s)v(\vec{p}, s') = 2p_0\delta_{ss'} \\ u^\dagger(\vec{p}, s)v(-\vec{p}, s') = 0\,, &\qquad v^\dagger(\vec{p}, s)u(-\vec{p}, s') = 0 \\ \bar{u}(\vec{p}, s)u(\vec{p}, s') = 2m\delta_{ss'}\,, &\qquad \bar{v}(\vec{p}, s)v(\vec{p}, s') = -2m\delta_{ss'} \\ \bar{u}(\vec{p}, s)v(\vec{p}, s') = 0\,, &\qquad \bar{v}(\vec{p}, s)u(\vec{p}, s') = 0\,, \end{aligned}$$

where $\bar{u} = u^\dagger\gamma^0$ and $\bar{v} = v^\dagger\gamma^0$, with

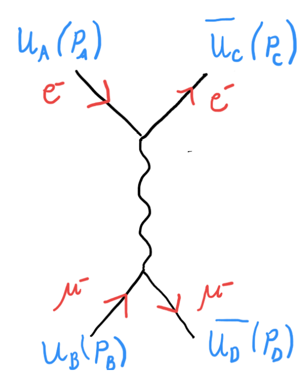
$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}\,.$$

(f) **[3 points]** By using the explicit form of the spinors $u(\vec{p}, s), v(\vec{p}, s)$, prove that they obey the following completeness relations (also known as *spin sums*):

$$\begin{aligned} \sum_{s=-\frac{1}{2}}^{s=+\frac{1}{2}} u_\alpha(\vec{p}, s)\bar{u}_\beta(\vec{p}, s) &= (\not{p} + m)_{\alpha\beta}\,, \\ \sum_{s=-\frac{1}{2}}^{s=+\frac{1}{2}} v_\alpha(\vec{p}, s)\bar{v}_\beta(\vec{p}, s) &= (\not{p} - m)_{\alpha\beta}\,, \end{aligned}$$

where α and β are indices in spin space.

2. **[8 points]** $e^-\mu^- \to e^-\mu^-$ **Scattering:** Consider the scattering $e^-\mu^- \to e^-\mu^-$, for which there is only one Feynman diagram:



(a) **[2 points]** Write down the scattering amplitude $\mathcal{M}$, and compute $|\mathcal{M}|^2$, leaving it in terms of the γ-matrices and $\not{p}$.

(b) **[2 points]** Experimentally, scattering amplitudes are not pure spin configurations, and often start from a mixture of spins (coming from an unpolarised beam), and the final states are not measured.

Therefore we sum over all final spins, in order to calculate the *unpolarised cross-section.* Recalling the γ-matrix relations determined in worksheet 3, show that we find the unpolarised cross-section

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{t^2}\left[2m_e^2 m_\mu^2 - m_\mu^2 p_A p_C - m_e^2 p_B p_D + (p_A \cdot p_B)(p_C \cdot p_D) + (p_A \cdot p_D)(p_B \cdot p_C)\right] ,$$

from

$$|\mathcal{M}|^2 \longrightarrow \overline{|\mathcal{M}|^2} = \frac{1}{(2s_A+1)(2s_B+1)} \sum_{\text{all spins}} |\mathcal{M}|^2 .$$

(c) **[3 points]** A short-form of the differential cross-section given in the lecture can be written,

$$d\sigma = \frac{|\mathcal{M}|^2}{F} dQ ,$$

where, for 2-particle scattering,

$$dQ = (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B) \frac{d^3 p_C}{(2\pi)^3 2p_C^0} \frac{d^3 p_D}{(2\pi)^3 2p_D^0}$$

is the Lorentz invariant phase space factor, and

$$F = 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} ,$$

is the flux factor. In the centre-of-mass frame, we have that $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D = 0$. Show that the differential cross-section in the centre-of-mass frame is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{COM}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \frac{p_f}{p_i} ,$$

with $|\vec{p}_A| = p_i$ and $|\vec{p}_C| = p_f$.
Hint: Start by writing F in terms of s, then introduce spherical co-ordinates to find an expression for dQ. Determine dQ by looking at $\int dQ$ and recognising that

$$\delta(f(p_f)) = \frac{1}{|f'(p_{f0})|}\delta(p_f - p_{f0}) .$$

(d) **[1 point]** Calculate the differential (unpolarised) cross-section for the $e^-\mu^- \to e^-\mu^-$ scattering process in the non-relativistic limit in the centre-of-mass frame. Reduce your answer as much as possible in terms of θ.