Einführung in Theoretische Teilchenphysik

Lectures: Prof. Dr. M. M. Mühlleitner – Exercises: M.Sc. Martin Gabelmann, Dr. Sophie Williamson

Exercise Sheet 6

Hand-in Deadline: Friday 18.12.20, 14:00.
Discussion: Tuesday 12.01.20, Thursday 14.01.20.

1. [**4 points**] **Mandelstam Variables**:

   Consider a general scattering process $A + B \to C + D$. The kinematics of this process can be determined by the Lorentz invariant Mandelstam variables:

   $$s = (p_A + p_B)^2 \,; \qquad t = (p_A - p_C)^2 \,; \qquad u = (p_A - p_D)^2 \,.$$

   where p_i is the 4-momentum of the respective particles.

   (a) [**1 point**] Prove the identity

   $$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \,,$$

   where m_i denote the masses of the corresponding particles.

   (b) [**1 point**] Derive the following relations in the centre of mass frame, in the case of identical particle masses ($m_a = m_b = m_c = m_d \equiv m$):

   $$s = 4(|\vec{p}_A|^2 + m^2) \,; \qquad t = -4|\vec{p}_A|^2 \sin^2\left(\frac{\theta}{2}\right) \,; \qquad u = -4|\vec{p}_A|^2 \cos^2\left(\frac{\theta}{2}\right) \quad ,$$

   with $\vec{p}_i$ the 3-momentum of the respective particles and θ the angle between particles A and C.

   (c) [**2 points**] Now consider Compton scattering,

   $$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k') \,.$$

   Determine expressions for the Mandelstam variables for this process in the laboratory frame as a function of the electron mass and the frequency of the photon.

2. [**5 points**] **Electron-Positron Annihilation**:

   Consider the process of electron-positron annihilation into two photons,

   $$e^- e^+ \to \gamma\gamma \,.$$

   (a) [**1 points**] Draw all Feynman diagrams contributing at leading order. Label them with the momenta of the internal propagators and the wave function factors for the external particles.

(b) **[2 points]** Use the Feynman rules of QED to write down the contribution of each diagram to the scattering amplitude. (Your result should be in the form $\bar{v}(p)\frac{1}{\not{p}+\not{k}-m}\ldots$, with no further simplifications necessary.)

Outline, without doing any explicit calculations, what needs to be done further to obtain a cross section for this process. State the factors needed for the final expression.

(c) **[2 points]** A correction to this process is given by the process with an additional photon in the final state, $e^-e^+ \to \gamma\gamma\gamma$. Draw the corresponding Feynman diagrams. Use your result to help you deduce how many diagrams there are for n photons in the final state.

3. **[7 points] Massive QED:** Consider the QED Lagrangian with one fermion field, ψ, and a photon field A_μ,

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where D_μ is the covariant derivative, and $F^{\mu\nu}$ is the field strength tensor,

$$D_\mu = \partial_\mu + ieA_\mu \ , \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ ,$$

and where the fields transform according to

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \ , \qquad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \, .$$

(a) **[3 points]** Show by explicit calculation that $\mathcal{L}_{\text{QED}}$ is gauge invariant, and show that this is contrary to the mass term, $\mathcal{L}_{\text{mass}} = \frac{m_A^2}{2}A^\mu A_\mu$, which is not.

(b) **[2 points]** Gauge invariance can be achieved by adding instead the term

$$\mathcal{L}_{\text{S}} = \frac{m_S^2}{2}\left(A^\mu + \frac{1}{m_S}\partial^\mu\sigma)(A_\mu + \frac{1}{m_S}\partial_\mu\sigma\right)$$

to the Lagrangian, where σ is an additional scalar field. How must $\sigma(x)$ transform so that $\mathcal{L}_{\text{S}}$ remains invariant?

(c) **[2 points]** For a complete theory we need the additional contribution,

$$\mathcal{L}_{\text{G}} = -\frac{1}{2\xi}(\partial^\mu A_\mu + m_S\xi\sigma)^2 \ , \quad \xi \in \mathbb{R} \, .$$

What condition needs to be placed on α so that the Lagrangian remains gauge invariant?

4. **[4 points] Lie Algebra Structure of SU(3)**:

A generic finite transformation $S \in SU(3)$ in the fundamental representation may be written in terms of 8 real parameters and the corresponding generators of the Lie algebra as

$$S = \exp\left(-\frac{i}{2}\sum_{i=1}^{8}\alpha_i\lambda_i\right) \quad \text{with} \quad \text{Tr}\left(\frac{\lambda_i}{2} \times \frac{\lambda_j}{2}\right) = \frac{1}{2}\delta_{ij}$$

where λ_i denote the 3×3 Gell-Mann matrices. The Gell-Mann matrices define the $SU(3)$ generators in the fundamental representation and are hermitian, traceless matrices satisfying:

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i\, f^{ab}_c\, \frac{\lambda_c}{2} \qquad \text{or equally} \qquad [T^a\, T^b] = i\, f^{ab}_c\, T^c \, . \tag{1}$$

(a) **[2 points]** Using Eq. (1), and recalling that the generators fulfill the Jacobi identity

$$[T^a\,,[T^b\,,T^c]] + [T^b\,,[T^c\,,T^a]] + [T^c\,,[T^a\,,T^b]] = 0\,, \tag{2}$$

show that $\mathrm{Tr}\,(T^a\,T^b) = C(F)\,\delta^{ab}$, where $C(F)$ is a (real) numerical factor. Evaluate $C(F)$ explicitly, using e.g. the first Gell-Mann matrix,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

(b) **[1 point]** Another important representation is the adjoint representation, for which the generators are given by the structure constants themselves,

$$[F_i(A)]_{jk} \;\equiv\; -i\,f_{ijk}\;,$$

and which satisfy

$$-(F_j\,F_i)_{mn} + (F_i\,F_j)_{mn} = i\,f_{ijk}\,(F_k)_{mn} \qquad \text{and therefore} \qquad [F_i, F_j]_{mn} = i\,f_{ijk}\,F_{kmn}\,.$$

Check that $\mathrm{Tr}[F_a(A)\,F_b(A)] = f_{ade}\,f_{bde} \equiv C(A)$, where again $C(A)$ is a real number.

(c) **[1 point]** Verify that the quadratic Casimir of the group, $F^2 = F^a\,F_a$, commutes with every individual generator F_a.