Karlsruher Institut für Technologie

Einführung in Theoretische Teilchenphysik

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Exercise Sheet 7

Hand-in Deadline: Friday 15.01.21, 14:00.
Discussion: Tuesday 19.01.21, Thursday 21.01.21.

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1. **[9 points] Goldstone bosons in O(3):**

   We consider a model with a scalar field, $\sigma = (\sigma_1,\, \sigma_2,\, \sigma_3)^T$, living in the fundamental representation of O(3). The Lagrangian of the model is given by

   $$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^\dagger (\partial^\mu \sigma) - \lambda(\sigma^2 + \mu^2)^2$$

   where μ^2 and λ are real parameters.

   (a) Calculate the mass spectrum for each of the following cases:

   i. **[1 point]** $\mu^2 > 0$,

   ii. **[2 points]** $\mu^2 < 0$. Start with the ansatz $\sigma \to \sigma' + (0, 0, v)^T$.

   Hint: The masses m_i can be expressed as coefficients of the quadratic term in the Lagrangian. Read off the terms $-\frac{m_i^2}{2}\sigma_i\sigma_i$ from $\mathcal{L}$.

   (b) Now we want to "calibrate" the model via an interaction with the vector field, W_μ^a, $(a = 1 \dots 3)$. For this, we replace the partial derivative in $\mathcal{L}$ with the covariant derivative,

   $$\partial_\mu \to D_\mu = \partial_\mu + i\, g\, t^a_{kl} W^a_\mu \ ,$$

   where $t^a_{kl} = -i\varepsilon_{akl}$ are the generators of the O(3) group.
   With this replacement, again calculate the masses of the scalar and vector bosons in the cases:

   i. **[1 point]** $\mu^2 > 0$,

   ii. **[5 points]** $\mu^2 < 0$. Here, use the fact that, in the ground state, σ can be expanded as $\sigma = e^{\frac{i}{v} t\, \Theta}(\sigma_0 + \eta')$, with $v = \sqrt{-\mu^2}$, $\sigma_0 = (0, 0, v')^T$, $\eta' = (0, 0, \eta)^T$ and $\Theta = (\theta_1, \theta_2, 0)$.

2. **[4 points] SU(N) groups:** The SU(N) group consists of unitary $N \times N$ matrices with a unit determinant. An N-dimensional vector, $\psi^T = (\psi_1, ..., \psi_N)$, transforms as

   $$\psi_i \to U_i^j \psi_j \ , \tag{1}$$

   where U is an element of SU(N). This is called the fundamental representation, **N**. In the anti-fundamental representation, $\overline{\mathbf{N}}$, the N-dimensional vector transforms as

   $$(\psi_i)^* \equiv \bar{\psi}^i \to (U^\dagger)^i_j \bar{\psi}^j \ .$$

(a) **[1 point]** Show that for $N = 3$, the object

$$\chi = \epsilon^{ijk} \psi_i^{(1)} \psi_j^{(2)} \psi_k^{(3)} ,$$

where $\psi^{(I)}, I = 1, 2, 3$ are fundamental representations of SU(3), is invariant. Note that the totally antisymmetric tensor, $\epsilon^{ijk}, \epsilon^{123} = 1$, is invariant under SU(3).

(b) **[2 points]** Show that the antisymmetric tensor, $A_{ij} = -A_{ji}$, $i, j = 1, ...N$ remains antisymmetric after an SU(N) transformation. Show the same for a symmetric tensor $S_{ij} = S_{ji}$. This means that $\mathbf{N} \times \mathbf{N}$ can be decomposed into a symmetric and antisymmetric part, which are both irreducible. Compute the dimension (i.e. the number of independent components) of A and S for SU(3).

(c) **[1 point]** Find the dimension of a totally symmetric 3-index tensor, S_{ijk}, for SU(3) and SU(5).

3. **[7 points] Transformation of the Covariant Derivative.**

The covariant derivative,

$$D_\mu = \partial_\mu + i\, g\, A_\mu = \partial_\mu + i\, g\, A_\mu^a T^a ,$$

is explicitly dependent on the chosen representation of the generators T^a of the gauge group. Let us consider a transformation of the covariant derivative and of the gauge field,

$$D'_\mu = U D_\mu U^{-1} , \qquad A'_\mu = U A_\mu U^{-1} - \frac{i}{g} U(\partial_\mu U^{-1}) ,$$

where the representation matrices $U = e^{i\theta^a T^a}$ are given in the fundamental representation. Using this, prove that the covariant derivative transforms like

$$D'_\mu = V D_\mu V^{-1} ,$$

for any arbitrary representation V, and calculate the transformation explicitly.

Hint: Use the Baker-Hausdorff formula,

$$e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} A_n ,$$

where $A_n = [B, A_{n-1}], A_0 = A$. Use the notation

$$V = e^{i\theta^a T^a} \equiv e^B , \qquad A \equiv \partial_\mu + igT^a A_\mu^a ,$$

where you may wish to treat the first and second terms in A separately (first using $A = \partial_\mu$, then $A = A_\mu$). Expand the first few terms of the Baker-Hausdorff formula and conclude that $V A_\mu V^{-1} = T^c U_{cb}^{\text{adj}} A_\mu^b$, and $V(\partial_\mu V^{-1}) = -\partial_\mu(i\theta^a) T^c W_{ac}^{\text{adj}}$ for any representation T^c. When considering the transformation of the partial derivative, act on a field ψ to get the full expression for the transformation. To simplify the expression for D'_μ further, rewrite the transformation of the vector field assuming $T^c = T^c_{\text{fund}}$, and use the fact that

$$\text{Tr}(T^a_{\text{fund}} T^b_{\text{fund}}) = T^a_{\text{fund, ij}} T^b_{\text{fund, ji}} = \frac{1}{2} \delta^{ab} .$$