Problem set 2

Submission deadline: 16 May, 16:00
Discussion of solutions: 17 May, 11:30

Problem 4: Second Friedmann equation

For the case of a spatially flat universe ($\kappa = 0$) the FLRW metric in cartesian coordinates is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a(t)^2 & 0 & 0 \\ 0 & 0 & -a(t)^2 & 0 \\ 0 & 0 & 0 & -a(t)^2 \end{pmatrix} . \tag{1}$$

a) Show that the spatial components of the Ricci tensor are given by

$$R_{ij} = (\ddot{a}a + 2\dot{a}^2)\delta_{ij} . \tag{2}$$

b) Obtain the spatial components of the Einstein equation.

c) Show that these equations are automatically satisfied if the first Friedmann equation holds and energy and momentum are conserved.

Problem 5: Particle horizon

Throughout this problem we assume negligible curvature ($\kappa = 0$). Consider a photon emitted at the moment of the Big Bang ($t = 0$).

a) Taking the point of emission to be $r = 0$, show that the trajectory of the photon must obey $\mathrm{d}r = \mathrm{d}\eta$, where η denotes conformal time.

b) Show that the physical distance of the photon from the origin at a later time t is given by

$$l_\mathrm{H} = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} . \tag{3}$$

c) Calculate l_H for a universe filled with non-relativistic matter and for a universe filled with radiation. Express your results in terms of the Hubble rate $H(t)$.

d) Show that for a universe filled with non-relativistic matter and vacuum energy (such that $\Omega_\mathrm{m} + \Omega_\Lambda = 1$), the function

$$a(t) = a_0 \left(\frac{\Omega_\mathrm{m}}{\Omega_\Lambda}\right)^{1/3} \left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t\right)\right]^{2/3} \tag{4}$$

solves the Friedmann equation.

e) Show that at time t_0, defined via the requirement $a(t_0) = a_0$, l_H is proportional to H_0^{-1}.

f) Use numerical integration to determine the constant of proportionality for $\Omega_\Lambda = 0.7$ and $\Omega_\mathrm{m} = 0.3$.

The quantity l_H is called the particle horizon. It corresponds to the size of the causally connected (i.e. observable) part of the universe.

Problem 6: Recollapse

Consider a universe with positive curvature ($\kappa = 1$) filled with non-relativistic matter.

a) Show that the Friedmann equation can be written as

$$H^2 = \frac{a_m}{a^3} - \frac{1}{a^2} \,, \tag{5}$$

where a_m parametrises the amount of matter contained in the universe.

b) Rewrite the Friedmann equation in terms of conformal time, i.e. replace $\dot{a}$ by $a' \equiv \mathrm{d}a/\mathrm{d}\eta$.

c) Show that the solution of the resulting equation is

$$a = a_m \sin^2 \frac{\eta}{2} \,, \tag{6}$$

i.e. the universe reaches its maximal size at $\eta = \pi$ and then collapses back to the singularity.

d) Find the total lifetime of the universe in terms of a_m.

e) Argue that a sufficient amount of dark energy can prevent a universe with negative curvature from recollapsing.