

Problem set 3

Submission deadline: 30 May, 16:00
 Discussion of solutions: 31 May, 11:30

Problem 7: Chemical potential (double credit)

- a) Show that for a relativistic fermion with $T \gg m$, the equilibrium number density can be written as

$$n = \frac{gT^3}{2\pi^2} \int_{-y}^{\infty} dq \frac{q^2 + 2yq + y^2}{e^q + 1}, \quad (1)$$

where $y = \mu/T$ and $q = x - y$ with $x = E/T$.

- b) Find the corresponding expression for the number density of anti-particles \bar{n} in terms of y and $r = x + y$.

- c) Show that

$$n - \bar{n} = \frac{gT^3}{6\pi^2} \left[\pi^2 \left(\frac{\mu}{T} \right) + \left(\frac{\mu}{T} \right)^3 \right]. \quad (2)$$

- d) Find the corresponding expression for bosons.

- e) Taking $\Omega_b = 0.05$ for the present-day abundance of baryons, assuming that all baryons are protons and that the universe is electrically neutral, calculate the present-day number density of electrons, $n_{e,0}$.

- f) Taking $g_{*,s} = 3.94$ and assuming that the present-day number density of positrons $\bar{n}_{e,0}$ is negligible, calculate $(n_{e,0} - \bar{n}_{e,0})/s_0$. Which special property makes this quantity particularly interesting?

- g) Calculate $n_e - \bar{n}_e$ at the beginning of Big Bang Nucleosynthesis ($T = 5 \text{ MeV}$).

- h) Determine the corresponding chemical potential of electrons.

Hint:

$$\int_0^{\infty} \frac{x}{e^x + 1} dx = \frac{\pi^2}{12}, \quad \int_0^{\infty} \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}. \quad (3)$$

Problem 8: Cannibal domination

Consider a non-relativistic particle species ϕ that does not interact with any other particle species in the universe. Let us consider the effect of number changing processes, such as $3\phi \rightarrow 2\phi$.

- a) Show that if these processes are in equilibrium, the chemical potential must vanish.
b) Neglecting the contribution from pressure, show that entropy conservation implies

$$\sqrt{\frac{T}{m}} e^{-m/T} a^3 = \bar{a}^3, \quad (4)$$

where \bar{a} is a constant.

- c) Show that for $m \gg T$ the solution of this equation is approximately

$$\frac{T}{m} \approx \frac{1}{3 \log(a/\bar{a})}. \quad (5)$$

- d) Show that the energy density of ϕ decreases more rapidly with increasing a than usual non-relativistic matter, but more slowly than radiation.

A particle species with number-changing processes is sometimes called “cannibalistic”: As the universe cools down the particles “eat” each other to “stay warm”, i.e. they convert rest mass into kinetic energy.