Problem set 5

Submission deadline: 27 June, 16:00
Discussion of solutions: 28 June, 11:30

Problem 12: Comoving gauge

In a general gauge the perturbed energy-momentum tensor is given by[1]

$$T^\mu_{\ \nu} = \begin{pmatrix} \bar\rho & 0 \\ 0 & -\bar P \delta^i_j \end{pmatrix} + \begin{pmatrix} \delta\rho & -(\bar\rho + \bar P)(v_i - B_i) \\ (\bar\rho + \bar P)v_i & -\delta P \delta^i_j \end{pmatrix} , \tag{1}$$

where $B_i = \partial_i B$ is a metric perturbation: $\delta g_{0i} = -B_i$.
Consider an infinitesimal coordinate transformation

$$X^\mu \to \tilde X^\mu = X^\mu + \xi^\mu(\eta, \mathbf{x}) \tag{2}$$

with $\xi^0 = T$ and $\xi^i = L^i$. Under this change of coordinates, the energy momentum tensor transforms as

$$T^\mu_{\ \nu}(X) = \frac{\partial X^\mu}{\partial \tilde X^\alpha}\frac{\partial \tilde X^\beta}{\partial X^\nu}\tilde T^\alpha_{\ \beta}(\tilde X) . \tag{3}$$

a) Show that the coordinate transformation leads to the following transformations of matter perturbations:

$$\begin{aligned} \delta\rho &\to \delta\rho - T\bar\rho' \\ \delta P &\to \delta P - T\bar P' \\ v_i &\to v_i - L_i' \end{aligned} \tag{4}$$

b) Show that the combination

$$\Delta \equiv \frac{\delta\rho}{\bar\rho} + \frac{\bar\rho'}{\bar\rho}(B - v) , \tag{5}$$

where $v_i = \partial_i v$, and $B_i = \partial_i B$ is a metric perturbation, is invariant under coordinate transformations.

c) Show that it is possible to choose a gauge, where $v = B = 0$, such that $\Delta = \delta = \frac{\delta\rho}{\bar\rho}$. This gauge is called "comoving gauge".

[1] For the derivation, see `https://www.mv.helsinki.fi/home/hkurkisu/CosPer.pdf`, section 9.

Problem 13: Horizon exit

Consider a flat ΛCDM universe with

$$H = H_0 \sqrt{\Omega_{\rm rad} \left(\frac{a_0}{a}\right)^4 + \Omega_{\rm m} \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda} \,. \tag{6}$$

a) Show that the conformal Hubble rate $\mathcal{H} = aH$ has a minimum for $a < a_0$. Find the corresponding value of a for $\Omega_{\rm m} = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_{\rm rad} \approx 0$ and $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}}$.

b) Argue that this means that there are perturbations that were subhorizon at some time in the past but are now superhorizon.

c) Find the smallest conformal momentum $k_{\rm min}$ of any perturbation that was subhorizon at some point in the past. Calculate the corresponding physical size of this perturbation in the present universe.

d) Compare your result to the physical size of perturbations that exit the horizon today.

Problem 14: Tensor perturbations

Let us consider tensor perturbations of the metric

$$ds^2 = a^2(\eta)\left[{\rm d}\eta^2 - (\delta_{ij} + \hat{E}_{ij}){\rm d}x^i{\rm d}x^j\right] \,, \tag{7}$$

where $\hat{E}_{ij}$ is a traceless and transverse tensor. Any such tensor can be decomposed into a sum of two different polarization states ($\alpha = +, \times$):

$$\hat{E}_{ij}(\eta, \mathbf{x}) = \sum_\alpha h_\alpha(\eta, \mathbf{x}) e^\alpha_{ij} \,. \tag{8}$$

The linearised Einstein tensor is given by $\delta G_{00} = 0$, $\delta G_{0i} = 0$ and

$$\delta G_{ij} = \frac{1}{2a^2}\left(\partial_\eta^2 \hat{E}_{ij} + 2\mathcal{H}\partial_\eta \hat{E}_{ij} - \partial_k\partial_k \hat{E}_{ij}\right) \,. \tag{9}$$

a) Assuming that there are no tensor perturbations in the matter, show that the Einstein equation implies
$$h''_\alpha(\eta, k) + 2\mathcal{H}h'_\alpha(\eta, k) + k^2 h_\alpha(\eta, k) = 0 \,. \tag{10}$$

b) Argue that the third term is negligible in the superhorizon regime, leading to the solution $h_\alpha = h_\alpha^{(i)}$.

c) Using the ansatz $h_\alpha(\eta, k) = f_\alpha(\eta, k)/a(\eta)$, show that an approximate solution in the subhorizon regime is given by

$$h_\alpha(\eta, k) = \frac{A}{a(\eta)}\cos(k\eta) + \frac{B}{a(\eta)}\sin(k\eta) \,. \tag{11}$$

d) Find the constants A and B by requiring that the superhorizon solution is recovered in the limit $\eta \to 0$ (you can assume radiation domination for small η).