Problem set 6

Submission deadline: 11 July, 15:30
Discussion of solutions: 11 July, 15:45

Problem 15: Evolution of dark matter perturbations

After photon decoupling, the equations of energy and momentum conservation for DM and baryons read

$$\delta_b' - k^2 v_b = 3\Phi' \qquad \delta_{\rm DM}' - k^2 v_{\rm DM} = 3\Phi' \tag{1}$$

$$v_b' + \frac{2}{\eta} v_b = -\Phi \qquad v_{\rm DM}' + \frac{2}{\eta} v_{\rm DM} = -\Phi \tag{2}$$

while the first Einstein equation gives

$$k^2\Phi = -\frac{6}{\eta^2}\left(\frac{\Omega_{\rm DM}}{\Omega_{\rm m}}\delta_{\rm DM} + \frac{\Omega_b}{\Omega_{\rm m}}\delta_b\right) . \tag{3}$$

a) Show that for a subhorizon mode these equations imply

$$\delta_{\rm DM}'' + \frac{2}{\eta}\delta_{\rm DM}' - \frac{6}{\eta^2}\delta_{\rm DM} = -\frac{6}{\eta^2}\frac{\Omega_b}{\Omega_{\rm m}}\Delta , \tag{4}$$

where $\Delta = \delta_{\rm DM} - \delta_b$.

b) Show that the general solution of this equation is given by

$$\delta_{\rm DM} = \frac{\Omega_b}{\Omega_m}\Delta + \alpha\eta^2 + \frac{\beta}{\eta^3} , \tag{5}$$

where α and β are numerical constants.

c) By matching to initial conditions (i.e. evaluating eqs. (1) and (2) at photon decoupling), show that

$$\alpha = \frac{1}{\eta_{\rm dec}^2}\left[\frac{\Omega_{\rm DM}}{\Omega_m}\delta_{\rm DM}^{\rm dec} + \frac{\Omega_b}{5\Omega_m}\left(3\delta_b^{\rm dec} + k^2\eta_{\rm dec}v_b^{\rm dec}\right)\right] \tag{6}$$

Hint: You can use the fact that dark matter perturbations are proportional to a during matter domination, such that

$$\left(\delta_{\rm DM}'\right)^{\rm dec} = \left(\frac{a'}{a}\delta_{\rm DM}\right)^{\rm dec} = \frac{2}{\eta_{\rm dec}}\delta_{\rm DM}^{\rm dec} . \tag{7}$$

Problem 16: Perturbed photon geodesics

Let $P^\mu = (P^0, P^i)$ denote the four-momentum of a photon. We define the three-momentum p via $p^2 = -g_{ij}P^iP^j$ and the requirement that p^i and P^i point in the same direction (given by the unit vector $\hat{p}^i$).

a) Show that for a perturbed metric in Newtonian gauge, the condition $P^\mu P_\mu = 0$ implies

$$P^0 = \frac{p}{a}(1-\Psi) \tag{8}$$

$$P^i = \frac{p\hat{p}^i}{a}(1+\Phi)\,. \tag{9}$$

b) Using the Christoffel symbols

$$\Gamma^0_{00} = \mathcal{H} + \frac{\partial\Psi}{\partial\eta} \tag{10}$$

$$\Gamma^0_{i0} = \frac{\partial\Psi}{\partial x^i} \tag{11}$$

$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left[\frac{\partial\Phi}{\partial\eta} + 2\mathcal{H}(\Phi+\Psi)\right]\delta_{ij} \tag{12}$$

show that

$$\Gamma^0_{\alpha\beta}P^\alpha P^\beta = -\frac{p^2}{a^2(1+2\Psi)}\left[-2\mathcal{H} + \frac{\partial\Phi}{\partial\eta} - \frac{\partial\Psi}{\partial\eta} - 2\hat{p}^i\frac{\partial\Psi}{\partial x^i}\right]\,. \tag{13}$$

c) Use this result to show that the geodesic equation

$$P^0\frac{\mathrm{d}P^0}{\mathrm{d}\eta} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta \tag{14}$$

becomes

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}\eta} = -\mathcal{H} - \hat{p}^i\frac{\partial\Psi}{\partial x^i} + \frac{\partial\Phi}{\partial\eta}\,. \tag{15}$$

Problem 17: Optical depth

After recombination the universe is neutral (and hence transparent to photons) until structure formation leads to reionization. As a result, CMB photons can scatter off free electrons, which changes their direction. The resulting mixing of photons from different points on the last-scattering surface reduces CMB anisotropies.

To estimate the magnitude of this effect, we can assume that reionization happens suddenly at $z = z_{\rm reio}$, and that all hydrogen atoms are ionized, while all helium atoms remain neutral:

$$n_e(z) = \begin{cases} n_H & z < z_{\rm reio} \\ 0 & z \geq z_{\rm reio} \end{cases} \tag{16}$$

The probability for a CMB photon to scatter is then given by $\exp(-\tau_{\rm reio})$, where

$$\tau_{\rm reio} = \sigma_{\rm T} \int_{t_{\rm dec}}^{t_0} {\rm d}t\, n_e(t) = \sigma_{\rm T} \int_0^{z_{\rm dec}} n_e(z) \frac{1}{(1+z)H(z)} {\rm d}z \tag{17}$$

with the Thomson cross section $\sigma_{\rm T} = 6.65 \times 10^{-25}\,{\rm cm}^2 = 1713\,{\rm GeV}^{-2}$.

a) Calculate $\tau_{\rm reio}$ for $z_{\rm reio} = 10$ using the approximate ΛCDM parameters $\Omega_m = 0.3$, $\Omega_b = 0.05$, $\Omega_\Lambda = 0.7$, $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}} = 1.4 \times 10^{-42}\,{\rm GeV}$.

   Hint: You can express n_H in terms of n_γ using the information extracted from Big Bang Nucleosynthesis.

b) Reionization cannot mix photons from regions of space that are not causally connected at reionization. Argue that this means that CMB multipoles $\ell < \ell_{\rm reio} = \pi\eta_0/\eta_{\rm reio}$ are not affected by reionization.

c) Calculate $\ell_{\rm reio}$.

The parameter z_{reio} constitutes the sixth (and final) independent parameter of the ΛCDM model.