Problem set 7

Submission deadline: 25 July, 16:00
Discussion of solutions: 26 July, 11:30

Problem 18: The Lee-Weinberg bound

We assume that in addition to the three well-known neutrino species there is a fourth neutrino ν with mass $m_\nu \gg 1\,\text{MeV}$ and interactions

$$\mathcal{L} = \frac{g}{2\cos\theta_\text{W}}\bar{\nu}\gamma_\mu(1-\gamma^5)\nu Z^\mu \,, \tag{1}$$

where $g \approx 0.65$ is the weak gauge coupling and $\theta_\text{W} \approx 0.50$ is the Weinberg angle. This interaction induces the annihilation processes $\nu\bar{\nu} \to f\bar{f}$, where f denotes Standard Model fermions. The corresponding cross section (assuming $m_\nu \ll M_Z$) is given by

$$\langle\sigma v\rangle = \frac{g^4\, m_\nu^2}{16\pi\, \cos^4\theta_\text{W}\, M_Z^4} \times C \,, \tag{2}$$

where $M_Z = 91\,\text{GeV}$ is the Z-boson mass and C is a numerical constant that depends on the number of fermions that are kinematically accessible. For the mass range of interest one finds $C \sim 6$.

a) Check explicitly that the additional neutrino would be in thermal equilibrium with the Standard Model particles for $T \approx m_\nu$. You can assume $g_* \approx 80$.

b) Estimate the relic abundance $\Omega_\nu h^2$ from thermal freeze-out.

c) To be consistent with observations, $\Omega_\nu h^2$ must not exceed the total amount of DM in the Universe, $\Omega_\text{DM} h^2 = 0.12$. Use this requirement to obtain a lower bound on m_ν.

Problem 19: The unitarity bound

It can be shown in a very general way from the requirement of unitarity (i.e. the conservation of probability) that the annihilation cross section for a Majorana fermion must satisfy the inequality

$$\sigma < \frac{\pi}{m_{\rm DM}^2 \, v_{\rm rel}^2} \,, \tag{3}$$

where $v_{\rm rel} = |\mathbf{v}_{\rm rel}|$ is the relative velocity between two dark matter particles, which follows the distribution

$$f(\mathbf{v}_{\rm rel}) = \left(\frac{x}{4\pi}\right)^{3/2} \exp\left(-\frac{x\, v_{\rm rel}^2}{4}\right) \tag{4}$$

with $x = m_{\rm DM}/T$.

a) Derive an upper bound on $\langle \sigma v_{\rm rel} \rangle$ as a function of $m_{\rm DM}$.

b) Calculate a lower bound on the thermal abundance of a dark matter particle with mass $m_{\rm DM}$. Combine this result with the measured value $\Omega_{\rm DM} h^2 = 0.12$ to derive the so-called unitarity bound on the WIMP mass.

Extra credit (5 marks): Derive eq. (4) starting from the assumption that the velocity distribution of each individual dark matter particle is given by the Boltzmann distribution $f(\mathbf{v}) \propto \exp(-E/T)$, where $E \approx m_{\rm DM}(1 + \frac{1}{2} v^2)$.

Problem 20: Gravitino dark matter

The gravitino is a well-motivated dark matter candidate that interacts only gravitationally. Dimensional analysis tells us that the cross section for gravitinos to interact with other types of particles is approximately given by

$$\langle \sigma v \rangle \sim \frac{1}{M_{\rm Pl}^2} \,. \tag{5}$$

a) Show that for temperatures $T \ll M_{\rm Pl}$ the interaction rate of gravitinos Γ is tiny compared to the Hubble rate and hence gravitinos never enter into thermal equilibrium with the SM.

b) Assuming that the initial abundance of gravitinos is negligible ($Y_{\tilde{G}} \equiv n_{\tilde{G}}/s \ll Y_{\tilde{G}}^{\rm eq}$), write down the Boltzmann equation for the evolution of the gravitino number density $n_{\tilde{G}}$. Show that this equation can be written as

$$\frac{\mathrm{d}Y_{\tilde{G}}}{\mathrm{d}T} = -\frac{\langle \sigma v \rangle \left(Y_{\tilde{G}}^{\rm eq}\right)^2 s}{H\, T} \,. \tag{6}$$

c) Show that if the gravitino mass is negligible, the right-hand side of this equation is independent of temperature and that hence the present-day value of $Y_{\tilde{G}}$ is directly proportional to the reheating temperature $T_{\rm R}$, which is the highest temperature ever reached in the early universe.

d) Using the observed value of the DM relic abundance, obtain an upper bound on $T_{\rm R}$ as a function of the gravitino mass $m_{\tilde{G}}$.

Hint: Cosmological observations give $s_0 = 2891\,\mathrm{cm}^{-3}$ and $\rho_c/h^2 = 1.05 \cdot 10^4\,\mathrm{eV cm}^{-3}$.