Problem set 4

1. Consider a Landau expansion of the free energy of the form

$$F = \frac{a}{2}m^2 + \frac{b}{4}m^4 + \frac{c}{6}m^6$$

with $c > 0$. Examine the phase diagram in the $a - b$ plane, and show that there is a line of critical transitions $a = 0$, $b > 0$ which joins a line of first order transitions $b = -4(ca/3)^{1/2}$ at a point $a = b = 0$ known as a tricritical point.

Supposing that a varies linearly with temperature and that b is independent of temperature, compare the value of the exponent β at the tricritical point with its value on the critical line.

From Yeomans, *Statistical Mechanics of Phase Transitions*

2. Consider a system with a modified expression for the Landau free energy, namely

$$\psi_h(t,m) = -hm + q(t) + r(t)m^2 + s(t)m^4 + u(t)m^6,$$

with $u(t)$ a fixed positive constant. Minimize ψ with respect to the variable m and examine the spontaneous magnetization m_0 as a function of the parameters r and s. In particular, show the following:[21]

(a) For $r > 0$ and $s > -(3ur)^{1/2}$, $m_0 = 0$ is the only real solution.

(b) For $r > 0$ and $-(4ur)^{1/2} < s \leq -(3ur)^{1/2}$, $m_0 = 0$ or $\pm m_1$, where $m_1^2 = \frac{\sqrt{(s^2-3ur)}-s}{3u}$. However, the minimum of ψ at $m_0 = 0$ is lower than the minima at $m_0 = \pm m_1$, so the ultimate equilibrium value of m_0 is 0.

(c) For $r > 0$ and $s = -(4ur)^{1/2}$, $m_0 = 0$ or $\pm(r/u)^{1/4}$. Now, the minimum of ψ at $m_0 = 0$ is of the same height as the ones at $m_0 = \pm(r/u)^{1/4}$, so a nonzero spontaneous magnetization is as likely to occur as the zero one.

[21] To fix ideas, it is helpful to use (r,s)-plane as our "parameter space."

(d) For $r > 0$ and $s < -(4ur)^{1/2}$, $m_0 = \pm m_1$ — which implies a *first-order* phase transition (because the two possible states available here differ by a *finite* amount in m). The line $s = -(4ur)^{1/2}$, with r positive, is generally referred to as a "line of first-order phase transitions."

(e) For $r = 0$ and $s < 0$, $m_0 = \pm(2|s|/3u)^{1/2}$.

(f) For $r < 0$, $m_0 = \pm m_1$ for all s. As $r \to 0$, $m_1 \to 0$ if s is positive.

(g) For $r = 0$ and $s > 0$, $m_0 = 0$ is only solution. Combining this result with (f), we conclude that the line $r = 0$, with s positive, is a "line of second-order phase transitions," for the two states available here differ by a *vanishing* amount in m.

The lines of first-order phase transitions and second-order phase transitions meet at the point $(r = 0, s = 0)$, which is commonly referred to as a *tricritical point* (Griffiths, 1970).

3. In the preceding problem, put $s = 0$ and approach the tricritical point along the r-axis, setting $r \approx r_1 t$. Show that the critical exponents pertaining to the tricritical point in this model are

$$\alpha = \frac{1}{2}, \beta = \frac{1}{4}, \gamma = 1, \text{ and } \delta = 5.$$

4. Arrott-plot method

Landau theory of a ferromagnet in a magnetic field $\vec{B}$ states that the free energy is given by

$$F(M) = F_0 + a(T - T_c)M^2 + bM^4 - MB$$

where a and b are constant and positive.

a) Show that

$$M^2 = u + v\frac{B}{M}$$

Follows, where u and v are constant.

b) How can this be used to determine T_c by plotting M^2 versus B/M for temperatures above, below and right at T_c. This method is known as the Arrott-plot method.

c) Determine the transition temperature T_c based on the data shown in Figure 1. What was the observation firstly reported in the corresponding publication [Dalichaouch et al., Phys. Rev. B **39**, 2423 (1989)]?

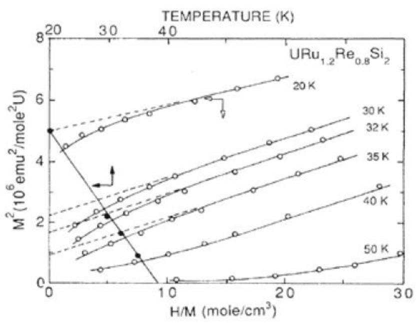

FIG. 4. Isotherms of M^2 vs H/M, where M is the magnetization and H is the applied magnetic field, for $URu_{1.2}Re_{0.8}Si_2$ for 20 K $\leq T \leq$ 50 K. Zero-field values of M^2, obtained by linear extrapolation of the high-field M^2 vs H/M data to $H=0$ (dashed lines), are plotted vs T. The Curie temperature Θ_C is defined as the temperature corresponding to $M^2=0$. Solid lines are guides to the eye.

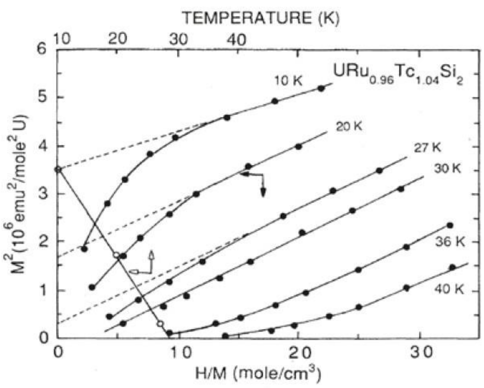

FIG. 5. Isotherms of M^2 vs H/M, where M is the magnetization and H is the applied magnetic field, for $URu_{0.96}Tc_{1.04}Si_2$ for 10 K $\leq T \leq$ 40 K. Zero-field values of M^2, obtained by linear extrapolation of the high-field M^2 vs H/M data to $H=0$ (dashed lines), are plotted vs T. The Curie temperature Θ_C is defined as the temperature corresponding to $M^2=0$. Solid lines are guides to the eye.

Figure 1:

Arrot plots of magnetization measurements in $URu_{2-x}M_xSi_2$ with M = Re and Tc. See details in the original figure captions above. Figure taken from Dalichaouch et al., Phys. Rev. B **39**, 2423 (1989).