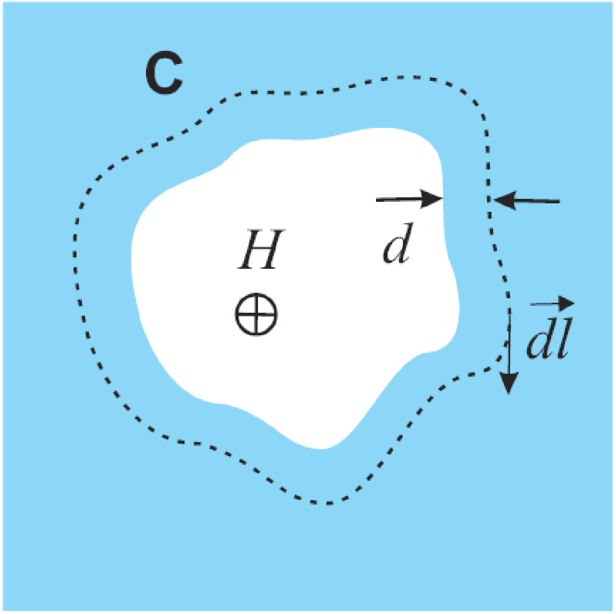
Superconductivity
Lecture 3



Electrodynamics of superconductors

- Two-fluid model
- The First London Equation
- The Second London Equation
- London Gauge
- Magnetic Field Penetration Depth
- Nonlocal electrodynamics
- Generalized London Equation
- Magnetic Flux Quantization

Fritz London (1900 -1954):

Fritz London was born on March 7, 1900 in Breslau, Germany (now Wroclaw, Poland). He was a German-American physicist who, with Walter Heitler, devised (1927) the first quantum mechanical treatment of the hydrogen molecule.

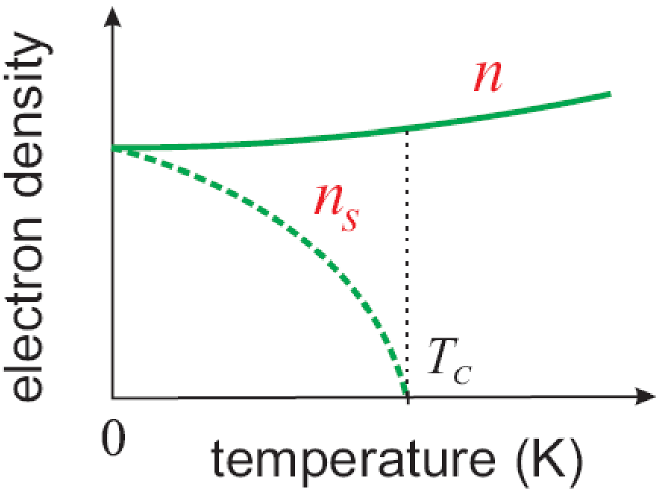
London was educated at the universities of Bonn, Frankfurt, Göttingen, Munich (Ph.D., 1921), and Paris. In 1939 he immigrated to the United States to become professor of theoretical chemistry at Duke University.

London's theory of the chemical binding of homopolar molecules is considered one of the most important advances in modern chemistry. With his brother, Heinz London, he developed (1935) the phenomenological theory of superconductivity.

Two-fluid model

Main assumptions of the two-fluid model:

- all free electrons of the superconductor are divided into two groups:
  - superconducting electrons of density n_s
  - normal electrons of density n_n.
- The total density of free electrons is
  $n = n_\mathrm{s} + n_\mathrm{n}$
- As the temperature increases from 0 to T_c, the density n_s decreases from n to 0.



The First London Equation

<u>We assume that:</u>

- both $\vec{E}$ and $\vec{H}$ are so weak that they do not have any appreciable influence on n_s.
- n_s = const everywhere.

The equation of motion for n_s electrons in an electric field $\vec{E}$ is

$$n_s m \frac{d\vec{v}_s}{dt} = n_s e \vec{E}$$

where m is the electron mass, e is the electron charge, $\vec{v}_s$ is the superfluid velocity.

Using $\vec{j}_s = n_s e \vec{v}_s$ we get

$$\boxed{\vec{E} = \frac{d}{dt}(\Lambda \vec{j}_s)}$$

where $\Lambda = m / n_s e^2$

The **first London equation** is simply Newton's second law for the superconducting electrons.

In the stationary state

$$d\vec{j}_s/dt = 0$$

we have

$$\vec{E} = 0$$

everywhere inside the superconductor.

The Second London Equation

Let us find a relation between $\vec{j}_\mathrm{s}$ and magnetic field $\vec{H}$ in a superconductor.

The kinetic energy density of the supercurrent is

$$W_\mathrm{kin} = n_\mathrm{s} m v_\mathrm{s}^2/2 = m j_\mathrm{s}^2/2 n_\mathrm{s} e^2$$

Taking into account Maxwell's equation $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j}_\mathrm{s}$

we get $W_\mathrm{kin} = \frac{\lambda^2}{8\pi} (\vec{\nabla} \times \vec{H})^2$ where $\lambda^2 = \frac{mc^2}{4\pi n_\mathrm{s} e^2}$

The free energy of the superconductor is

$$\mathcal{F}_{\mathrm{s}H} = \mathcal{F}_{\mathrm{s}0} + \frac{1}{8\pi} \int [\vec{H}^2 + \lambda^2 (\vec{\nabla} \times \vec{H})^2] \, \mathrm{d}V$$

magnetic energy density

free energy at zero magnetic field

The Second London Equation

$$\mathcal{F}_{sH} = \mathcal{F}_{s0} + \frac{1}{8\pi}\int [\vec{H}^2 + \lambda^2 (\vec{\nabla}\times \vec{H})^2]\, \mathrm{d}V$$

Let us now find the value of $\vec{H}$ corresponding to the minimum of $\mathcal{F}_{sH}$.

Assuming a small variation $\vec{H} \longrightarrow \vec{H} + \delta\vec{H}$ we find a change in $\mathcal{F}_{sH}$ as

$$\delta\mathcal{F}_{sH} = \frac{1}{8\pi}\int (2\vec{H}\,\delta\vec{H} + 2\lambda^2\vec{\nabla}\times \vec{H} \cdot \vec{\nabla}\times \delta\vec{H})\, \mathrm{d}V$$

The searched function $\vec{H}$ corresponds to a minimum of $\mathcal{F}_{sH}$, that is,

$$\delta\mathcal{F}_{sH} = 0$$

For an arbitrary $\delta\vec{H}$ we have to satisfy

$$\boxed{\vec{H} + \lambda^2\vec{\nabla}\times \vec{\nabla}\times \vec{H} = 0}$$

This is the **second London equation**.

London gauge

We can rewrite the second London equation as

$$\vec{j}_s = -\frac{c}{4\pi\lambda^2}\vec{A}$$

$$\text{div}\,\vec{A} = 0$$

$$\vec{A}\,\vec{n} = 0$$

the continuity of current and absence of a supercurrent source

no supercurrent can pass through the boundary of a superconductor

This equation is correct only in the so-called London gauge choice of the vector potential

We can rewrite the above equation as

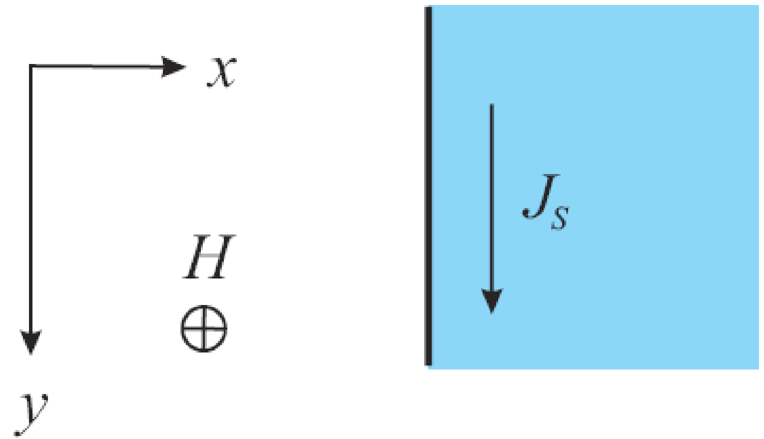
$$\boxed{\vec{j}_s = -\frac{1}{c\Lambda}\vec{A}}$$

alternative form of the 2nd London equation

$$\Lambda = 4\pi\lambda^2/c^2$$

Magnetic Field Penetration Depth

- The surface of the superconductor coincides with the plane $x = 0$
- An external magnetic field H_0 is oriented along the z axis
- To solve this problem, we shall use the second London equation



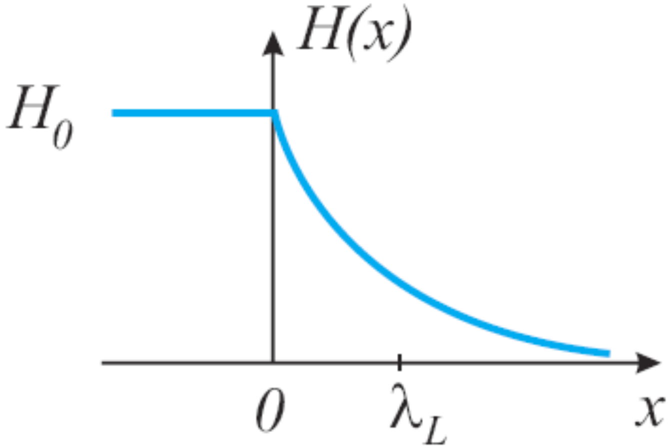
superconductor

Taking into account the symmetry of the problem we can write

$$\left.\begin{aligned} \mathrm{d}^2 H/\mathrm{d}x^2 - \lambda^{-2} H &= 0 \\ H(0) &= H_0 \\ H(\infty) &= 0 \end{aligned}\right\} \qquad H = H_0\,\mathrm{e}^{-x/\lambda}$$

Magnetic Field Penetration Depth

$$H = H_0\, \mathrm{e}^{-x/\lambda}$$



$$j_\mathrm{s} = (c/4\pi)\,\mathrm{d}H/\mathrm{d}x \quad \Rightarrow \quad j_\mathrm{s} = \frac{cH_0}{4\pi\lambda}\,\mathrm{e}^{-x/\lambda}$$

The length λ (often noted as λ_L) is called the **London penetration depth**:

$$\lambda = \left(\frac{mc^2}{4\pi n_\mathrm{s} e^2}\right)^{1/2}$$

Magnetic Field Penetration Depth

The length λ is temperature-dependent as it depends on n_s. A rather good approximation is given by the empirical formula

$$\lambda(T) = \frac{\lambda(0)}{\left(1 - (T/T_c)^4\right)^{1/2}}$$

Element	Al	Nb (crystal)	Nb (film)	Pb	Sn	YBCO
$\lambda(0)$, Å	500	470	900	390	510	1700

Nonlocal Electrodynamics of Superconductors

Everything said so far about the electrodynamics of superconductors falls into the category of the so-called *local electrodynamics*.

The supercurrent density $\vec{j}_{\mathrm{s}}$ is defined by the vector potential $\vec{A}$ at the same point only if the size of a Cooper pair ξ_0 is smaller than λ, which is the characteristic variation length of the vector potential.

For pure metals, however, the size of a pair ξ_0~ 1 μm and λ ~ 10-100 nm. Thus, local electrodynamics can not be applied in this case.

A nonlocal relation proposed by Pippard (1953):

$$\vec{j}_{\mathrm{s}}(\vec{r}) = \int \hat{Q}(\vec{r} - \vec{r}\,')\,\vec{A}(\vec{r}\,')\,\mathrm{d}\vec{r}\,'$$

Local case corresponds to $\hat{Q}(\vec{r} - \vec{r}\,') = \delta\left(\vec{r} - \vec{r}\,'\right)$.

Pippard's nonlocal kernel

$$\vec{j}_{\rm s}(\vec{r}) = \int \hat{Q}(\vec{r}-\vec{r}')\,\vec{A}(\vec{r}')\,{\rm d}\vec{r}'$$

$$\hat{Q}(\vec{r}-\vec{r}')\,\vec{A}(\vec{r}') = -\frac{3n_{\rm s}e^2}{4\pi mc\xi_0}\,\frac{(\vec{r}-\vec{r}')}{(\vec{r}-\vec{r}')^4}\,\left[\vec{A}(\vec{r}')\cdot\vec{r}-\vec{r}'\right]{\rm e}^{-|\vec{r}-\vec{r}'|/\xi_0}$$

The magnetic field penetration in the nonlocal case is non-exponential and penetration depth is defined as

$$\lambda = \frac{1}{H_0}\int_0^\infty H\,{\rm d}x$$

The true field dependence $H(x)$ can be approximated by an exponent with a new penetration depth

$$\lambda_{\rm P} \approx (\lambda^2\xi_0)^{1/3}$$

Generalized London Equation

Let us assume at this point that

- The supercurrent is carried by pairs of electrons called the Cooper pairs.
- Their elementary charge is $2e$.
- All pairs occupy the same energy level, i.e. form a condensate.
- The wavefunction of a particle from the condensate can be written as

$\Psi(\vec{r}) = (n_{\rm s}/2)^{1/2}\, {\rm e}^{{\rm i}\theta(\vec{r})}$ where θ is the phase of the wavefunction.

Consider a particle of mass $2m$ and charge $2e$ moving in a magnetic field. Its momentum can be written as

$$\hbar\nabla\theta = 2m\vec{v}_{\rm s} + \frac{2e}{c}\vec{A}$$

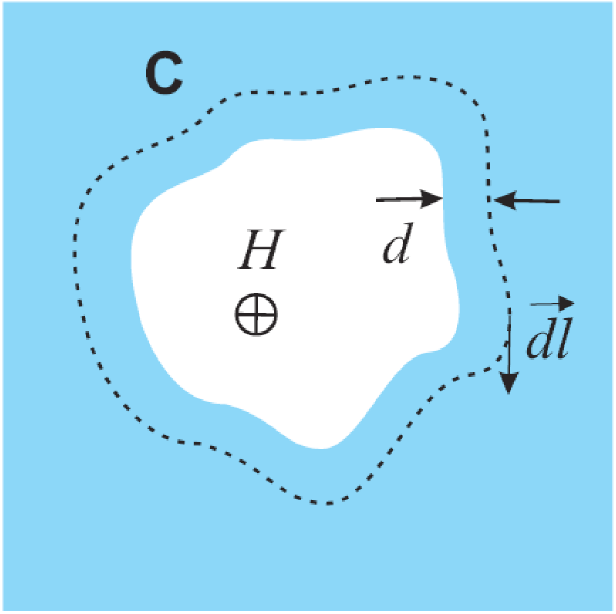
The generalized second London equation:

$$\vec{j}_{\rm s} = \frac{1}{c\Lambda}\left(\frac{\Phi_0}{2\pi}\nabla\theta - \vec{A}\right)$$

Magnetic Flux Quantization

Let us consider a hole inside a bulk superconductor.

- At $T > T_c$ the superconductor is in the normal state.
- We apply magnetic field H_0 and decrease the temperature so that the sample goes into the superconducting state.
- Some frozen magnetic flux will remain.
- This flux will be produced by the supercurrent flowing at the surface of the hole.



Magnetic Flux Quantization

$$\frac{\Phi_0}{2\pi}\oint_C \nabla\theta \, \mathrm{d}\vec{l} = \oint_C \vec{A} \, \mathrm{d}\vec{l}$$

$$\Phi = \oint_C \vec{A} \, \mathrm{d}\vec{l} = n\Phi_0$$

where

$$\Phi_0 = \frac{\pi\hbar c}{e} = \frac{hc}{2e} = 2.07 \times 10^{-7} \ \mathrm{G\,cm^2}$$

In SI units the same expression is written as

$$\Phi_0 = \frac{h}{2e} = 2.07 \times 10^{-15} \ \mathrm{V\,s}$$

Kinetic inductance

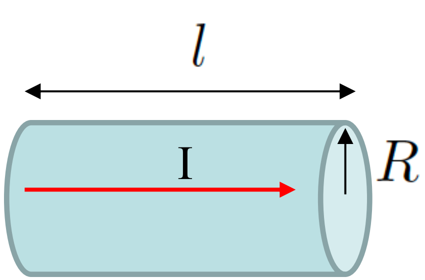
The *geometrical* inductance of a section of an electric circuit is defined by the energy of a magnetic field:

$$\mathcal{F}^{\mathrm{M}} = \frac{1}{8\pi}\int H^2\,\mathrm{dV} = \frac{1}{2\mathrm{c}^2}\,\mathrm{L}^{\mathrm{M}}\mathrm{I}^2$$

The current I in the circuit requires a part of the energy to be converted into the kinetic energy of the current carriers (electrons):

$$\mathcal{F}^{\mathrm{K}} = \int n\,\frac{mv^2}{2}\,\mathrm{dV} = \frac{1}{2\mathrm{c}^2}\,\mathrm{L}^{\mathrm{K}}\mathrm{I}^2$$

supercurrent density $\boldsymbol{j}_{\mathrm{s}} = n_{\mathrm{s}} e \boldsymbol{v}_{\mathrm{s}}$ ⇨ kinetic inductance $L^{\mathrm{K}} = c^2 \Lambda \int {j_{\mathrm{s}}}^2\,\mathrm{dV}/\mathrm{I}^2$

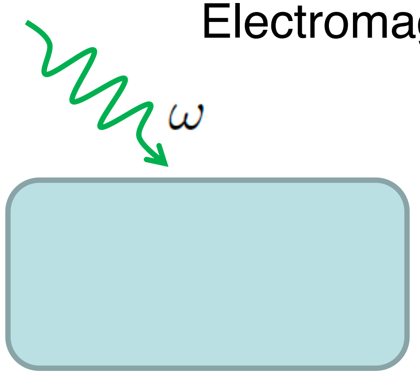


$$L^{\mathrm{K}} = l\lambda/R$$

inductance per square
$$L^{\mathrm{K}}_{\square} = 2\pi\lambda$$
$$L^{\mathrm{M}}_{\square} = 2\pi\lambda$$

total ⇨ $L_{\square} = 4\pi\lambda$

Complex Conductivity of a Superconductor



Electromagnetic wave with frequency ω

superconducting electrons $\boldsymbol{E} = \Lambda \, \mathrm{d}\boldsymbol{j}_\mathrm{s}/\mathrm{dt}$

normal electrons $e\boldsymbol{E} - \dfrac{m}{n_\mathrm{n} e} \dfrac{\boldsymbol{j}_\mathrm{n}}{\tau} = \dfrac{m}{n_\mathrm{n} e} \dfrac{\mathrm{d}\boldsymbol{j}_\mathrm{n}}{\mathrm{dt}}$

asuming $\boldsymbol{j}_\mathrm{s} \sim \mathrm{e}^{\mathrm{i}\omega\mathrm{t}}$ ⇨ $\boldsymbol{j}_\mathrm{n} = \dfrac{n_\mathrm{n}}{n_\mathrm{s}} \dfrac{\tau}{\Lambda} \dfrac{1 - \mathrm{i}\omega\tau}{1 + (\omega\tau)^2} \boldsymbol{E}$

The total current density $\boldsymbol{j} = \boldsymbol{j}_\mathrm{s} + \boldsymbol{j}_\mathrm{n}$ ⇨ $\boldsymbol{j} = \sigma \boldsymbol{E}\,, \quad \sigma = \sigma_1 - \mathrm{i}\sigma_2\,,$

$$\sigma_1 = \frac{n_\mathrm{n}}{n_\mathrm{s}} \frac{\tau}{\Lambda} \frac{1}{1 + (\omega\tau)^2}\,, \quad \sigma_2 = \frac{1}{\Lambda\omega} \left[1 + \frac{n_\mathrm{n}}{n_\mathrm{s}} \frac{(\omega\tau)^2}{1 + (\omega\tau)^2} \right]$$