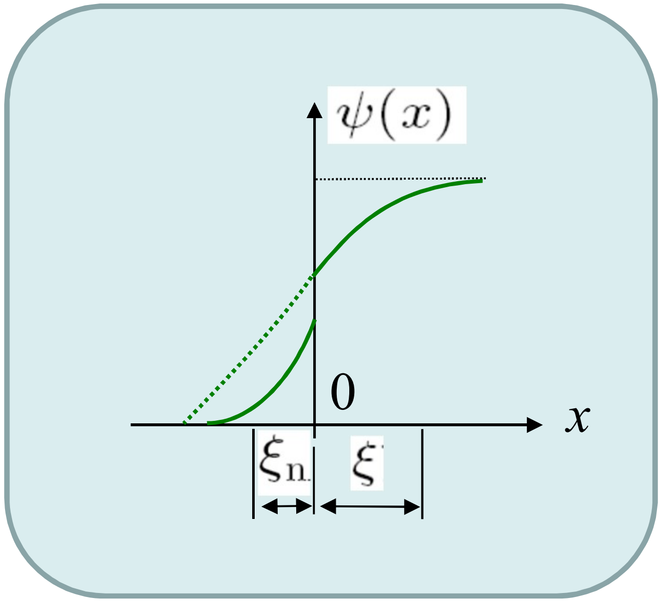
Superconductivity
Lecture 4



The Ginzburg-Landau Theory

- Introduction
- Second-order phase transitions
- Wavefunction of superconducting electrons
- Free energy density
- Ginzburg-Landau (GL) equations
- Boundary conditions
- Coherence length and penetration depth
- Proximity effect

Introduction

- The London theory does not take into account quantum effects

- The first quantum *phenomenological* theory of superconductivity was the **Ginzburg-Landau theory**
    - The superconducting state is more ordered than the normal one
    - The superconducting transition is a second order phase transition
    - Order parameter for a superconductor is nonzero at $T < T_c$

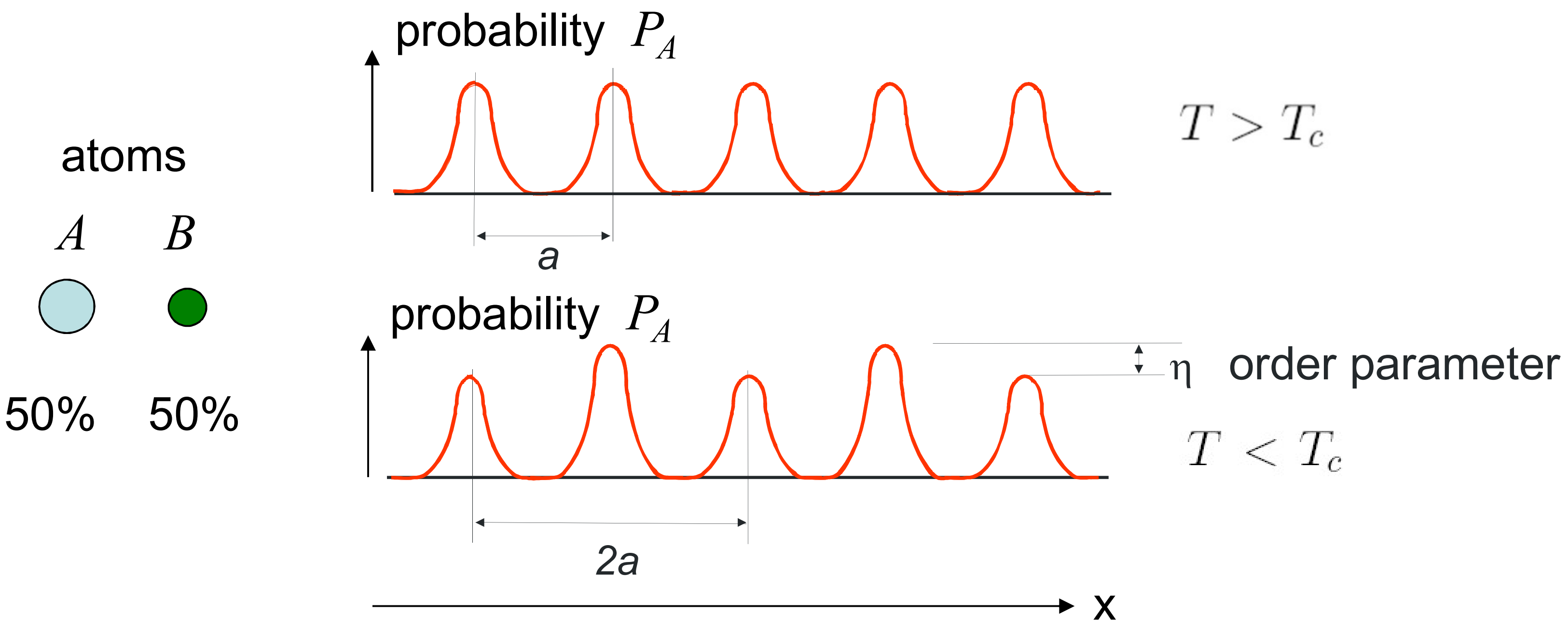
- Wavefunction of superconducting electrons $\Psi(\vec{r})$

- $\Psi(\vec{r})$ can be taken as order parameter

Second-order phase transitions

Examples:

- The ferromagnetic transition at the Curie point
- The transition of helium to the superfluid state
- Order-disorder transitions



Free energy

$\Psi(\vec{r})$ is the order parameter

$\Psi(\vec{r})$ is normalized to the density of the Cooper pairs: $|\Psi(\vec{r})|^2 = n_s/2$

Consider a homogeneous superconductor at $\vec{H} = 0$

- Ψ does not depend on $\vec{r}$
- expansion of the free energy in powers of $|\Psi_0|^2$ near T_c

$$F_{s0} = F_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4$$

F_{s0} is the free energy density of the superconductor

F_n is its free energy density in the normal state

α, β are phenomenological expansion coefficients characterising the material

First order approximation

F_{s0} at minimum: $\dfrac{\mathrm{d}F_{s0}}{\mathrm{d}|\Psi|^2} = 0 \Rightarrow |\Psi_0|^2 = -\alpha/\beta$

$$F_n - F_{s0} = \alpha^2/2\beta = H_{cm}^2/8\pi \Rightarrow H_{cm}^2 = 4\pi\alpha^2/\beta$$

Temperature dependence in the first order approximation in $(T_c - T)$:

$$\left.\begin{array}{l} \alpha = 0 \text{ at } T = T_c \\ \alpha < 0 \text{ at } T < T_c \end{array}\right\} \Rightarrow \alpha \sim (T - T_c)$$

$\beta > 0$ both at $T < T_c$ and at $T > T_c$ $\Rightarrow$ $\beta = \mathrm{const}$

Gibbs free energy density

A superconductor in a uniform external magnetic field $\vec{H}_0$

Gibbs free energy: $G = F - \dfrac{\vec{B}\vec{H}_0}{4\pi}$

the exact microscopic field at a given point of the superconductor

$$G_{sH} = G_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 +$$

$$+ \underbrace{\frac{1}{2m^*}\left|-i\hbar\nabla\Psi - \frac{2e}{c}\vec{A}\Psi\right|^2}_{\text{the kinetic energy density of the superconducting electrons}} + \underbrace{\frac{H^2}{8\pi} - \frac{\vec{H}\vec{H}_0}{4\pi}}_{\frac{(\vec{\nabla}\times\vec{A})^2}{8\pi} - \frac{(\vec{\nabla}\times\vec{A})\cdot\vec{H}_0}{4\pi}}$$

Let us find equations for functions $\Psi(\vec{r})$ and $\vec{A}(\vec{r})$

Variation δ_{Ψ^*} of Gibbs free energy density

Let us solve the variational problem with respect to $\Psi^*(\vec{r})$: $\delta_{\Psi^*} G_{sH} = 0$

$$\delta_{\Psi^*} G_{sH} = \int \mathrm{d}V \left[\alpha \Psi \, \delta\Psi^* + \beta \Psi \, |\Psi|^2 \, \delta\Psi^* + \right.$$

$$\left. + \frac{1}{4m} \left(\mathrm{i}\hbar \nabla \, \delta\Psi^* - \frac{2e}{c} \vec{A} \, \delta\Psi^* \right) \underbrace{\left(-\mathrm{i}\hbar \nabla \Psi - \frac{2e}{c} \vec{A} \, \Psi \right)}_{} \right] \quad ;$$

$$\Rightarrow \quad \delta_{\Psi^*} G_{sH} = \int \mathrm{d}V \left[\alpha \Psi + \beta \Psi \, |\Psi|^2 + \right.$$

$$\left. + \frac{1}{4m} \left(-\mathrm{i}\hbar \nabla - \frac{2e}{c} \vec{A} \right)^2 \Psi \right] \delta\Psi^* +$$

$$+ \oint_S \left[-\mathrm{i}\hbar \nabla \Psi - \frac{2e}{c} \vec{A} \, \Psi \right] \delta\Psi^* \, \mathrm{d}S = 0 \quad .$$

The first Ginzburg-Landau equation

For an arbitrary function $\delta\Psi^*$, both expressions in […] must be zero.

We obtain the first equation of the Ginzburg-Landau (GL) theory

$$\boxed{\alpha\Psi + \beta\Psi\left|\Psi\right|^2 + \frac{1}{4m}\left(i\hbar\nabla + \frac{2e}{c}\vec{A}\right)^2 \Psi = 0} \qquad (1)$$

and the boundary condition for it

$$\boxed{\left(i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\,\Psi\right)\vec{n} = 0} \qquad (2)$$

where $\vec{n}$ is the unit vector along the normal to the surface of a superconductor.

Variation $\delta_{\vec{A}}$ of Gibbs free energy density

Let us minimize the expression for G_{sH} with respect to $\vec{A}$:

$$\delta_{\vec{A}} G_{sH} = \int \mathrm{d}V \left\{ \frac{1}{4m} \delta_{\vec{A}} \left[\left(\mathrm{i}\hbar \nabla \Psi^* - \frac{2e}{c} \vec{A} \Psi^* \right) \right.\right.$$

$$\left.\left. \cdot \left(-\mathrm{i}\hbar \nabla \Psi - \frac{2e}{c} \vec{A} \Psi \right) \right] + \frac{1}{4\pi} \left(\vec{\nabla} \times \vec{A} \right) \cdot \left(\vec{\nabla} \times \delta \vec{A} \right) - \right.$$

$$\left. - \frac{\vec{H}_0}{4\pi} \cdot \left(\vec{\nabla} \times \delta \vec{A} \right) \right\}$$

; after elementary modifications

$$\delta_{\vec{A}} G_{sH} = \int \left[\frac{\mathrm{i}\hbar e}{2mc} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) + \right.$$

$$\left. + \frac{2e^2}{mc^2} \vec{A} \left| \Psi \right|^2 + \frac{1}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \vec{A} \right] \delta \vec{A} \, \mathrm{d}V = 0 \quad .$$

The second Ginzburg-Landau equation

$$\begin{aligned}\delta_{\vec{A}} G_{sH} &= \int \left[\frac{i\hbar e}{2mc} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) + \right. \\ &+ \left. \frac{2e^2}{mc^2} \vec{A} \left| \Psi \right|^2 + \frac{1}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \vec{A} \right] \delta \vec{A} \, \mathrm{d}V = 0\end{aligned}$$

For an arbitrary $\delta \vec{A}$, the above expression in […] must be zero.

As $\vec{H} = \vec{\nabla} \times \vec{A}$ and from Maxwell´s equation $\vec{j}_s = \frac{c}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \vec{A}$,

we obtain the second equation of the Ginzburg-Landau (GL) theory

$$\boxed{\vec{j}_s = -\frac{i\hbar e}{2m} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) - \frac{2e^2}{mc} \left| \Psi \right|^2 \vec{A}}$$

Normalized form of GL equations

Let us go over to a dimensionless wavefunction $\psi(\vec{r}) = \Psi(\vec{r})/\Psi_0$

where $\Psi_0{}^2 = n_s/2 = |\alpha|/\beta$

We will introduce two new notations: $\xi^2 = \dfrac{\hbar^2}{4m|\alpha|}$; $\lambda^2 = \dfrac{mc^2}{4\pi n_s e^2} = \dfrac{mc^2\beta}{8\pi e^2|\alpha|}$

1st equation

$$\boxed{\xi^2 \left(\mathrm{i}\nabla + \frac{2\pi}{\Phi_0}\vec{A} \right)^2 \psi - \psi + \psi\,|\psi|^2 = 0}$$

2nd equation

$$\boxed{\vec{\nabla} \times \vec{\nabla} \times \vec{A} = -\mathrm{i}\frac{\Phi_0}{4\pi\lambda^2}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{|\psi|^2}{\lambda^2}\vec{A}}$$

where $\Phi_0 = \dfrac{\pi\hbar c}{e}$ is the magnetic flux quantum.

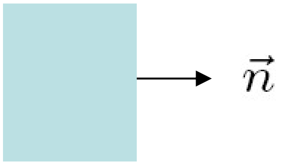
Boundary conditions

If ψ is written as $\psi = |\psi|\, \mathrm{e}^{\mathrm{i}\theta}$, the 2nd GL equation becomes

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \frac{|\psi|^2}{\lambda^2} \left(\frac{\Phi_0}{2\pi} \nabla\theta - \vec{A} \right)$$

The boundary condition of no supercurrent flowing through the surface:

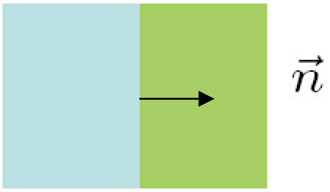
$$\left(\mathrm{i}\nabla + \frac{2\pi}{\Phi_0} \vec{A} \right) \vec{n}\psi = 0$$



where $\vec{n}$ is the unit vector along the normal to the surface of a superconductor.

Note: The boundary condition
for an interface with a normal metal:

$$\left(\mathrm{i}\nabla + \frac{2\pi}{\Phi_0} \vec{A} \right) \vec{n}\psi = \mathrm{i}a\psi \quad ,$$



where a is an arbitrary real number.

Gauge invariance of the GL theory

It is well known that a choice of $\vec{A}$ is equivocal. The transformation $\vec{A} = \vec{A}' + \nabla\varphi$ does not change the magnetic field:

$$\vec{H} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'$$

a single-valued scalar function

$\varphi(\vec{r})$

⇓

$$\vec{\nabla} \times \vec{\nabla}\varphi = 0$$

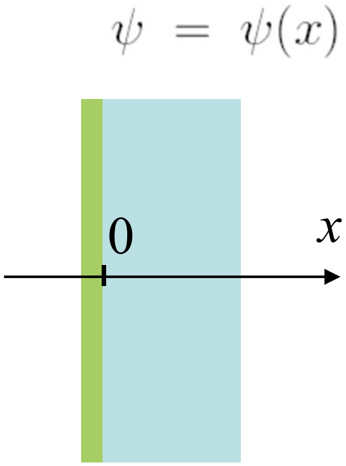
To make results of theoretical calculations independent of the choice of $\vec{A}$ (make them gauge invariant) we can use

$$\boxed{\begin{gathered} \vec{A} = \vec{A}' + \nabla\varphi \\ \psi = \psi' \exp\left[\mathrm{i}\,\frac{2\pi}{\Phi_0}\,\varphi(\vec{r})\right] \end{gathered}}$$

One can verify the gauge invariance of both GL equations, e.g.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}' = \frac{|\psi'|^2}{\lambda^2}\left(\frac{\Phi_0}{2\pi}\nabla\theta' - \vec{A}'\right)$$

Coherence length and penetration depth



The 1st GL equation: $-\xi^2 \, d^2\psi/dx^2 - \psi + \psi^3 = 0$ (3)

For a thin normal layer: $\psi = 1 - \varepsilon(x)$, $\varepsilon(x) \ll 1$.

$$\Rightarrow \quad \xi^2 \, d^2\varepsilon(x)/dx^2 - 2\varepsilon(x) = 0 \ .$$

Since $\varepsilon(\infty) = 0$ $\Rightarrow$ $\varepsilon = \varepsilon(0)\, e^{-\sqrt{2}\,x/\xi}$.

ξ is the characteristic scale of variation of the order parameter ψ . This length ξ is called the coherence length.

Recall: $\xi^2 = \dfrac{\hbar^2}{4m|\alpha|}$; $\lambda^2 = \dfrac{mc^2}{4\pi n_s e^2} = \dfrac{mc^2\beta}{8\pi e^2|\alpha|}$

$\Rightarrow$ $\lambda \sim (T_c - T)^{-1/2}$, $\xi \sim (T_c - T)^{-1/2}$.

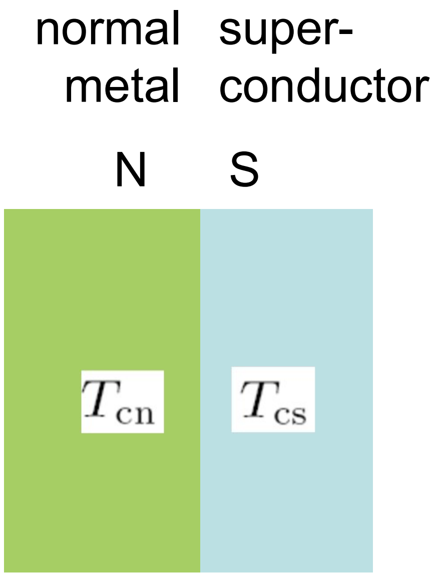
penetration depth coherence length

The GL parameter $\boxed{\kappa = \lambda/\xi}$

Proximity effect at NS interface

The role of the coherence length becomes evident when we consider clean interface between a normal metal N and a superconductor S.

- The Cooper pairs can penetrate from S into N and “live” there for some time.
- A thin N-layer becomes superconducting.
- The penetration of the Cooper pairs from S into N results in their reduced density in S.

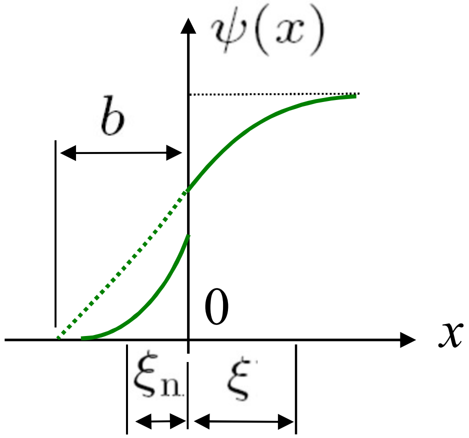


The behavior of the order parameter in the S region ($x > 0$) can be determined by solving Eq. (3)

$$-\xi^2(\mathrm{d}\psi/\mathrm{d}x)^2 - \psi^2 + \frac{1}{2}\psi^4 = C$$

At $x \rightarrow \infty$ we have $(\mathrm{d}\psi/\mathrm{d}x) \rightarrow 0$ and $\psi \rightarrow 1$

$\Rightarrow$ $C = -1/2.$ $\Rightarrow$ $\psi = \tanh\left[(x - x_0)/\sqrt{2}\xi\right]$

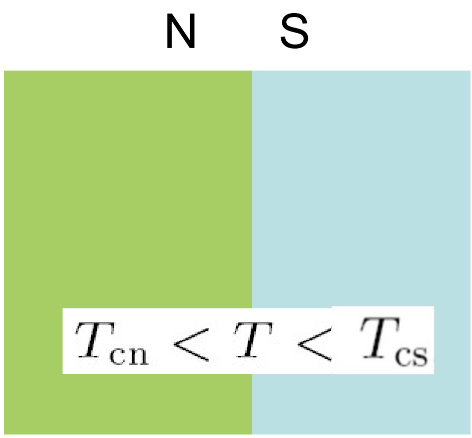


Order parameter in N

$$\psi = \tanh\left[(x - x_0)/\sqrt{2}\xi\right]$$

Here the constant x_0 is to be determined from the boundary condition at $x = 0$.

$$\frac{1}{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{1}{b} \Rightarrow -\sinh\left(\sqrt{2}\,\frac{x_0}{\xi}\right) = \sqrt{2}\,\frac{b}{\xi}\ .$$

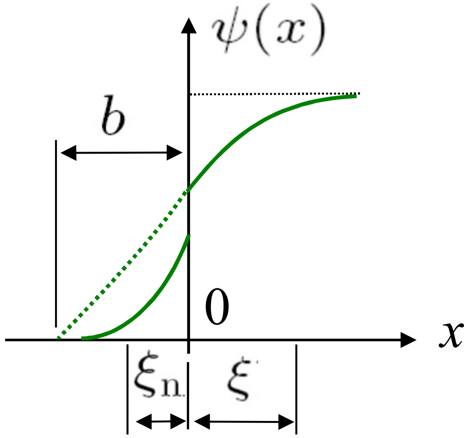


The behavior of the order parameter in the N region ($x < 0$) can be determined by solving Eq. (3).

$\alpha_{\mathrm{n}} \sim (T - T_{\mathrm{cn}})$, so that $\alpha_{\mathrm{n}} < 0$ at $T < T_{\mathrm{cn}}$ and $\alpha_{\mathrm{n}} > 0$ at $T > T_{\mathrm{cn}}$. For $T - T_{\mathrm{cn}} \ll T_{\mathrm{cn}}$ we get

$$\boxed{-{\xi_{\mathrm{n}}}^2(\mathrm{d}^2\psi/\mathrm{d}x^2) + \psi + \psi^3 = 0}$$

where ${\xi_{\mathrm{n}}}^2 = \hbar^2/4m\alpha_{\mathrm{n}}$.



For $\psi \ll 1$ using $\psi \rightarrow 0$ at $x \rightarrow \infty$ we get $\psi = \psi_0 \exp(-|x|/\xi_{\mathrm{n}})$.

Coherence length in N

The order parameter penetrates the N region and decays there exponentially over the characteristic length ξ_{n} .

Microscopic theory calculation gives the following results:

<u>"Clean" case</u>: when the electron mean free path $l_{\mathrm{n}} \gg \xi_{\mathrm{n}}$

$$\xi_{\mathrm{n}} = \frac{\hbar v_{\mathrm{Fn}}}{2\pi k_{\mathrm{B}} T}$$

<u>"Dirty" case</u>: when $l_{\mathrm{n}} \ll \xi_{\mathrm{n}}$

$$\xi_{\mathrm{n}} = \left(\frac{\hbar v_{\mathrm{Fn}} l_{\mathrm{n}}}{6\pi k_{\mathrm{B}} T} \right)^{1/2}$$

⇒ ξ_{n} ~ 0.1 – 1 μm