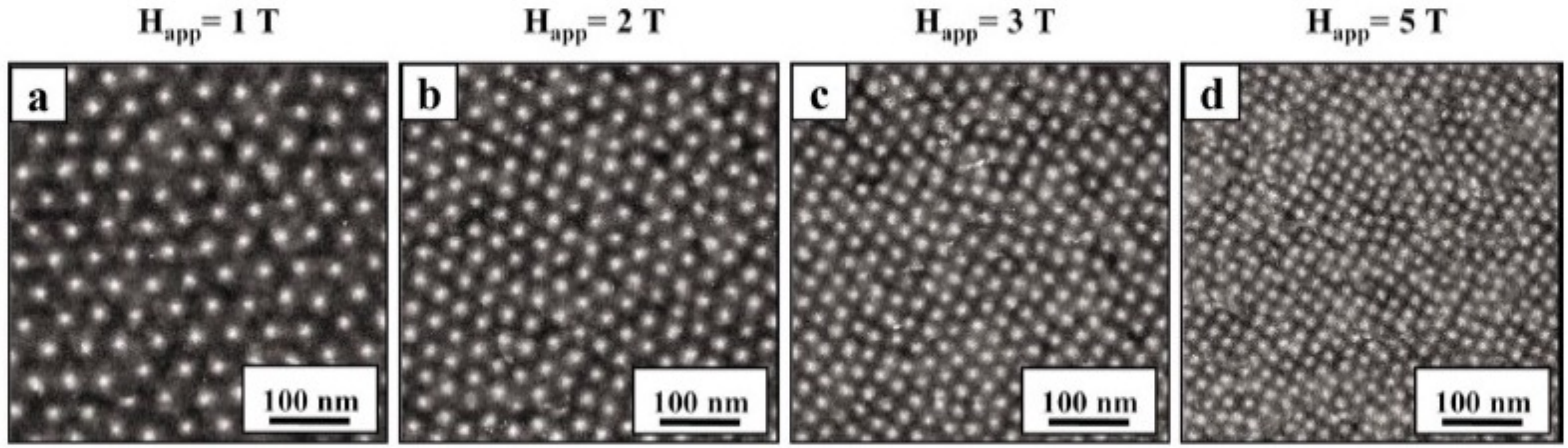
# Superconductivity
# Lecture 5



# Type I and type II superconductors. Vortices

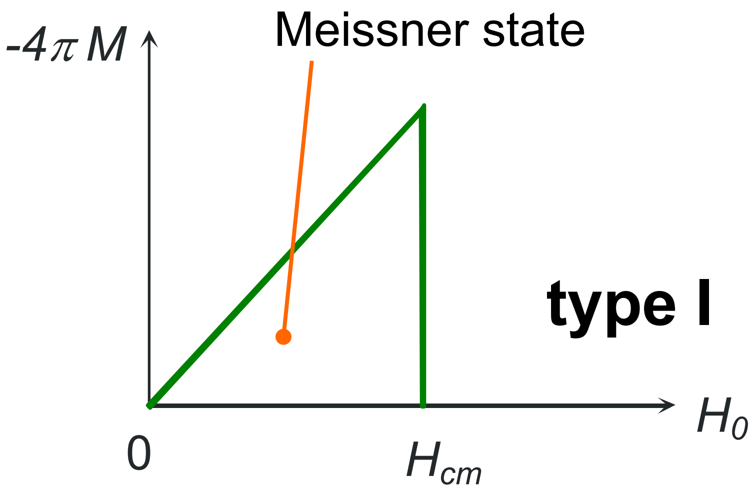
- Energy of NS interface
- Characteristic lengths $\lambda$ and $\xi$
- Extreme cases
- Vortex lattice
- Experimental methods of vortex imaging
- Magnetic field of a vortex
- Interaction between vortices

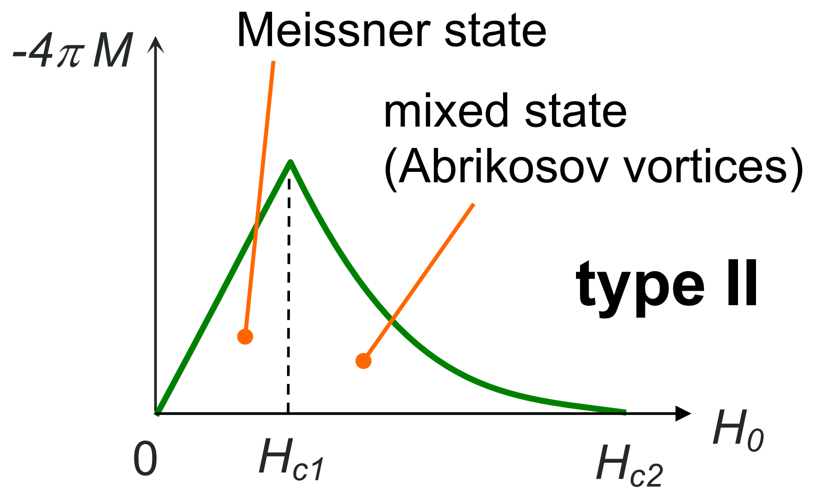
# Type I and type II superconductors

- Type-I and type-II superconductors show different responses to external magnetic field.

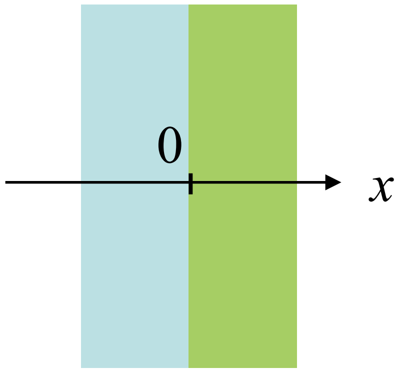
- The reason is that the surface energy of an interface between a normal metal and a superconducting region, $\sigma_{ns}$
  - for type-I $\sigma_{ns} > 0$
  - for type-II $\sigma_{ns} < 0$

- Roughly speaking,
  - for type-I $\lambda < \xi$
  - for type-II $\lambda > \xi$



# Interface normal metal – superconductor




S super-conductor | N normal metal

$$\psi = \psi(x)$$

$$\vec{H} = (0,\ 0,\ H(x))$$

$$\vec{A} = (0,\ A(x),\ 0)$$

Ginzburg-Landau (GL) equations

$$-\xi^2 \frac{d^2\psi}{dx^2} + \left(\frac{2\pi\xi}{\Phi_0}\right)^2 A^2\psi - \psi + \psi^3 = 0$$

$$\frac{d^2 A}{dx^2} = \left(\frac{\psi^2}{\lambda^2}\right) A$$

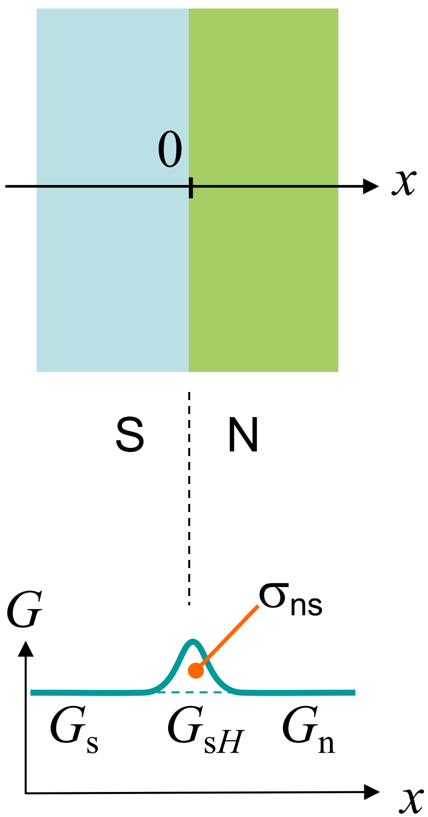
First integral $\left[1 - \left(\frac{2\pi\xi A}{\Phi_0}\right)^2\right]\psi^2 - \frac{1}{2}\psi^4 +$

$$+\left(\frac{2\pi\lambda\xi}{\Phi_0}\right)^2 \left(\frac{dA}{dx}\right)^2 + \xi^2\left(\frac{d\psi}{dx}\right)^2 = C$$

at $x \to -\infty$, $\psi \to 1$, $d\psi/dx \to 0$, and $A \to 0$ ⇨ $C = 1/2$

(*) $$\left[\left(\frac{2\pi\xi A}{\Phi_0}\right)^2 - 1\right]\psi^2 + \frac{1}{2}\psi^4 = \xi^2\left(\frac{d\psi}{dx}\right)^2 + \frac{H^2}{2H_{cm}^2} - \frac{1}{2}\,.$$

# NS interface: Gibbs free energy




Gibbs energy density

Energy of the interface $$\sigma_{\rm ns} = \int\limits_{-\infty}^{\infty} (G_{{\rm s}H} - G_{\rm n})\,{\rm d}x$$

Gibbs energy density $$G_{{\rm s}H} = F_{{\rm s}H} - H\,H_{\rm cm}/4\pi$$
$$G_{\rm n} = F_{\rm n} - H_{\rm cm}^2/8\pi$$

$$F_{{\rm s}H} = F_{\rm n} + \frac{H^2}{8\pi} + \frac{H_{\rm cm}^2}{4\pi}\left[-|\psi|^2 + \frac{1}{2}|\psi|^4 + \xi^2\left|{\rm i}\nabla\psi + \frac{2\pi}{\Phi_0}\vec{A}\psi\right|^2\right]$$

$$\Rightarrow \quad \sigma_{\rm ns} = \int\limits_{-\infty}^{\infty}\left\{\frac{H_{\rm cm}^2}{4\pi}\left[-\psi^2 + \frac{1}{2}\psi^4 + \xi^2\left(\frac{{\rm d}\psi}{{\rm d}x}\right)^2 + \left(\frac{2\pi\xi A}{\Phi_0}\right)^2\psi^2\right] + \frac{H^2}{8\pi} - \frac{H\,H_{\rm cm}}{4\pi} + \frac{H_{\rm cm}^2}{8\pi}\right\}{\rm d}x$$

# NS interface energy

Energy of the interface: $$\sigma_{\rm ns} = \frac{H_{\rm cm}^2}{2\pi} \int_{-\infty}^{\infty} \Bigg[ \underbrace{\xi^2 \left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2}_{>\,0} + \underbrace{\frac{H(H - H_{\rm cm})}{2H_{\rm cm}^2}}_{<\,0} \Bigg] \mathrm{d}x$$

As $\xi^2(\mathrm{d}\psi/\mathrm{d}x)^2 \sim 1$ the 1st term $$\int_{-\infty}^{\infty} \xi^2 \left(\mathrm{d}\psi/\mathrm{d}x\right)^2 \mathrm{d}x \sim \xi$$

the 2nd term $$\int_{-\infty}^{\infty} \frac{H(H - H_{\rm cm})}{2H_{\rm cm}^2} \mathrm{d}x \sim -\lambda$$

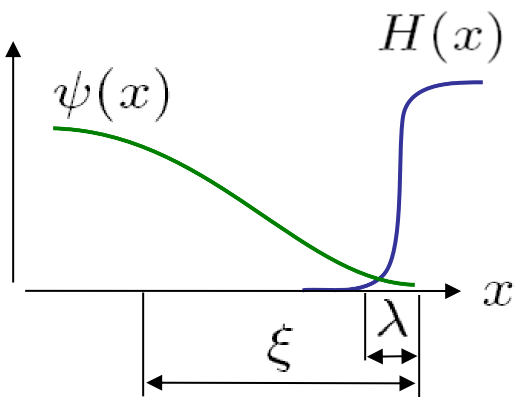
$\kappa = \lambda/\xi$

**Two limiting cases:**

| | |
|---|---|
| $\kappa \ll 1$, i.e., $\lambda \ll \xi$ | $\kappa \gg 1$, i.e., $\lambda \gg \xi$ |
| ⇨ $\sigma_{\rm ns} \sim H_{\rm cm}^2 \xi > 0$ | ⇨ $\sigma_{\rm ns} \sim -H_{\rm cm}^2 \lambda < 0$ |

# Type I and type II: Characteristic lengths

**type I**




extreme case

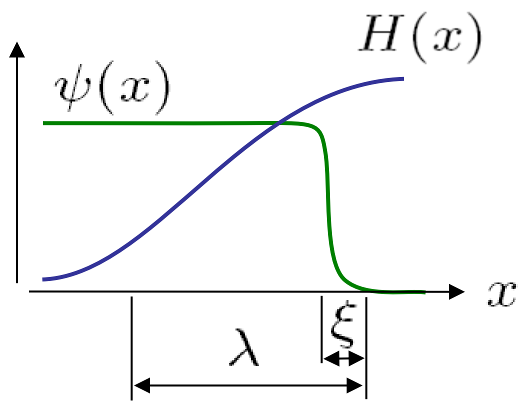
$\kappa \ll 1$, i.e., $\lambda \ll \xi$

exact calculation

$\kappa < 1/\sqrt{2}$
$\sigma_{\rm ns} > 0$

$$\sigma_{\rm ns} = 1.89\,\frac{H_{\rm cm}^2}{8\pi}\,\xi$$

**type II**




extreme case

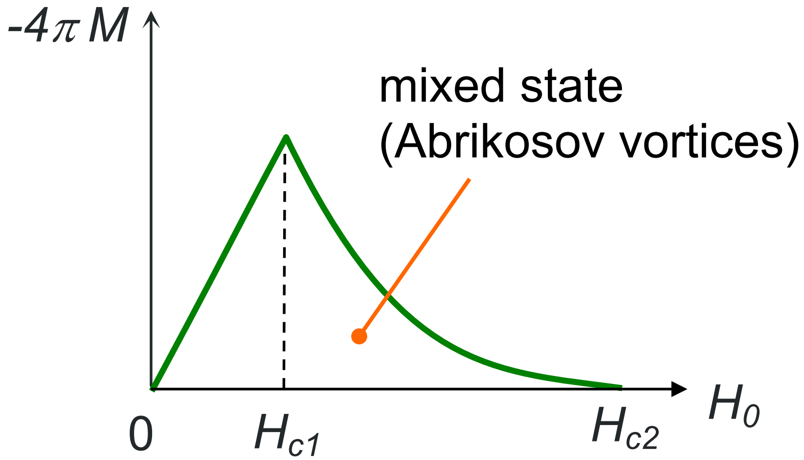
$\kappa \gg 1$, i.e., $\lambda \gg \xi$

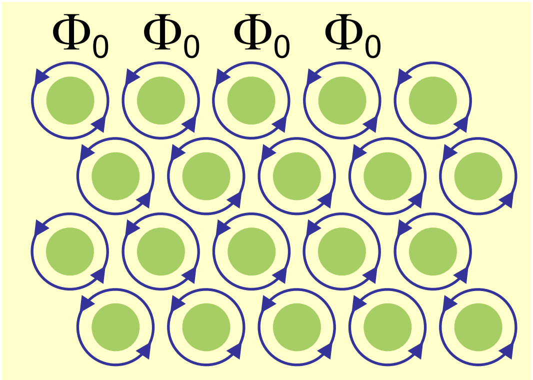
exact calculation

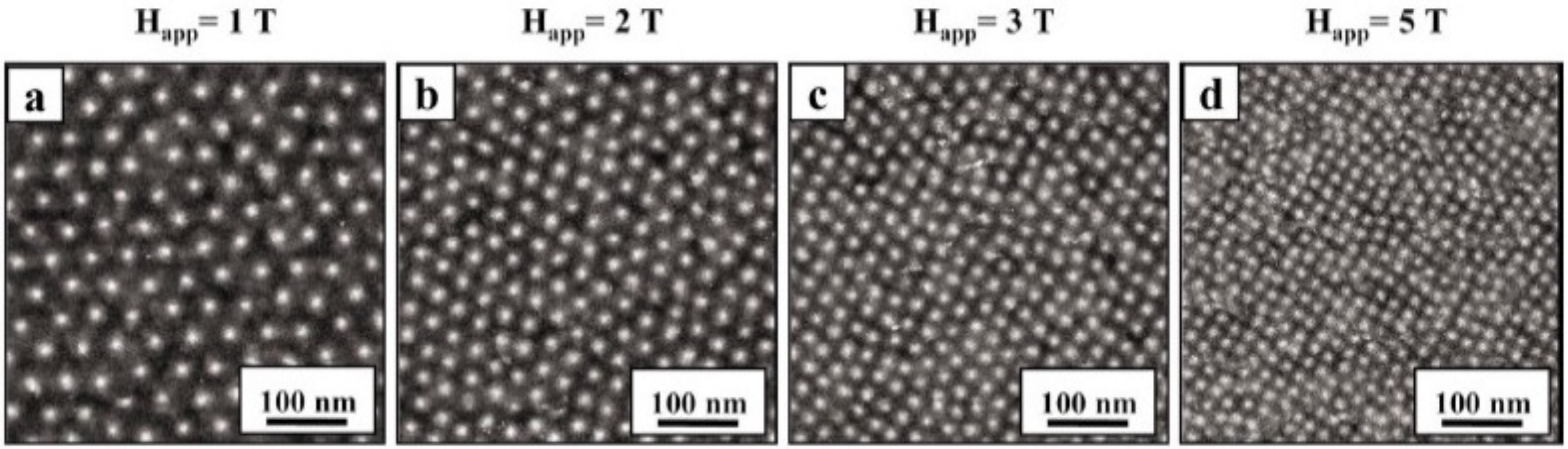
$\kappa > 1/\sqrt{2}$
$\sigma_{\rm ns} < 0$

$$\sigma_{\rm ns} = -\frac{H_{\rm cm}^2}{8\pi}\,\lambda$$

# Vortices in type II superconductors



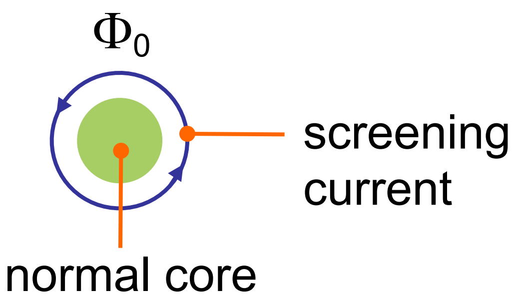



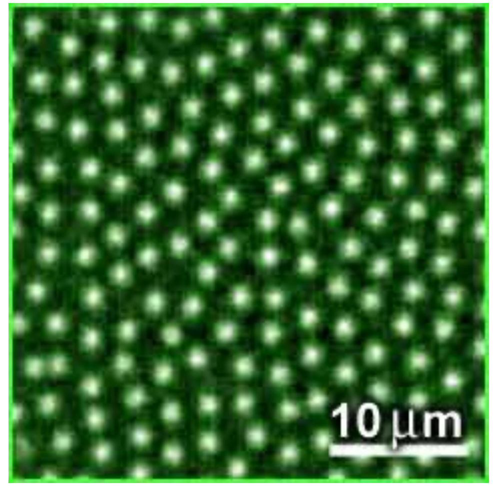

STM images obtained at NIST

NIST National Institute of Standards and Technology

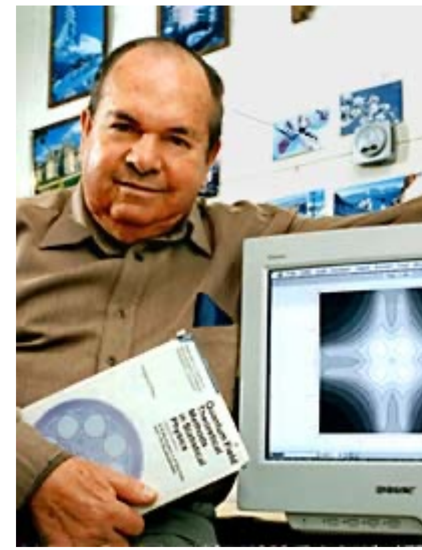
# Abrikosov vortices



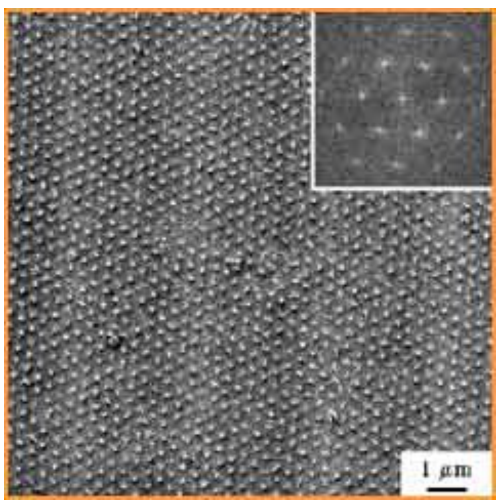

theoretical prediction 1957




Magneto-Optical Imaging
$NbSe_2$ crystal, B=3 G
P.E. Goa et al. University of Oslo

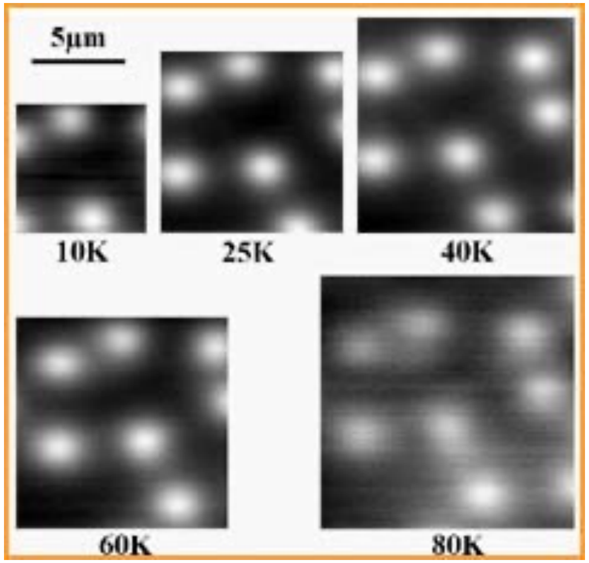


A. A. Abrikosov
(Nobel Prize 2003)

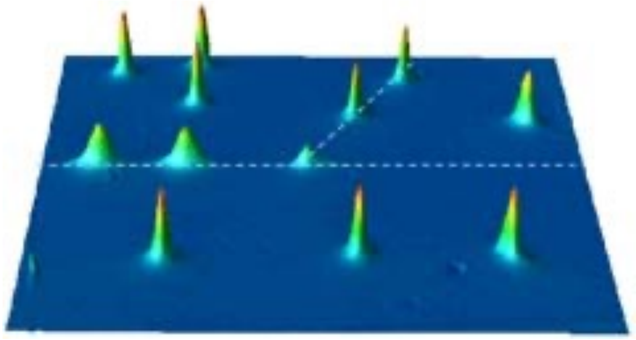



Decoration by ferromagnetic nano-particles, $MgB_2$ crystal, B=200 G
L. Ya. Vinnikov et al. ISSP Chernogolovka

# Magnetic imaging of individual vortices

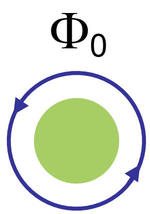



Scanning Hall probe
YBaCuO film, B=1000 G
A. Oral et al. University of Bath



Scanning SQUID microscopy of
half-integer vortex, 1996
YBaCuO grown on tricrystal substrate
J. R. Kirtley et al. IBM

# Magnetic field of a single vortex




We assume that GL parameter $\kappa \gg 1$, i.e., $\lambda \gg \xi$
and consider $r \gg \xi$ where $|\psi|^2 = 1$.

The 2nd GL equation $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \frac{1}{\lambda^2}\left(\frac{\Phi_0}{2\pi}\nabla\theta - \vec{A}\right)$

$$\Rightarrow \quad \vec{\nabla}\times\vec{H} = \frac{1}{\lambda^2}\left(\frac{\Phi_0}{2\pi}\nabla\theta - \vec{A}\right) \quad \Rightarrow \quad \vec{H} + \lambda^2\vec{\nabla}\times\vec{\nabla}\times\vec{H} = \frac{\Phi_0}{2\pi}\vec{\nabla}\times\nabla\theta \ .$$

At any point of the vortex, apart from its center, $\vec{\nabla}\times\nabla\theta = 0$.

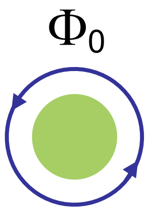
The center of the vortex represents a singularity, where $|\nabla\theta| \to \infty$.

By the Stokes theorem, $\int_{\odot} \vec{\nabla}\times\nabla\theta \, \mathrm{d}\vec{S} = \oint \nabla\theta \, \mathrm{d}\vec{l} = 2\pi$

(recall that each vortex carries one flux quantum)

$\Rightarrow \quad \vec{\nabla}\times\nabla\theta = 2\pi\delta(\vec{r})\vec{e}_{\mathrm{v}}$ , where $\vec{e}_{\mathrm{v}}$ is the unit vector along the vortex.

# Magnetic field of a single vortex



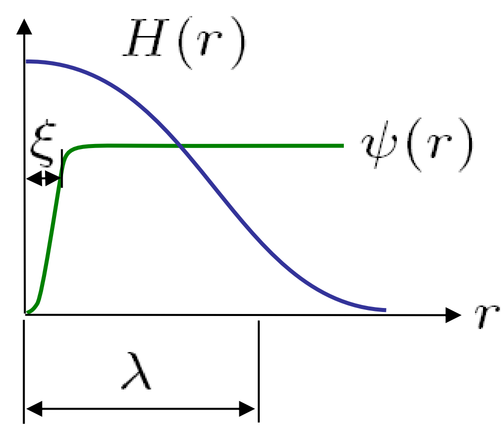

Finally we get an equation $\vec{H} + \lambda^2 \vec{\nabla} \times \vec{\nabla} \times \vec{H} = \Phi_0 \, \delta(\vec{r}) \, \vec{e}_{\mathrm{v}}$

subject to the boundary condition $H(\infty) = 0.$

The solution is $H = \dfrac{\Phi_0}{2\pi\lambda^2} K_0(r/\lambda)$, where $K_0$ is the MacDonald function.

$$K_0(z) \sim \begin{cases} \ln(1/z) & \text{at } z \ll 1\,, \\ e^{-z}/z^{1/2} & \text{at } z \gg 1\,. \end{cases}$$

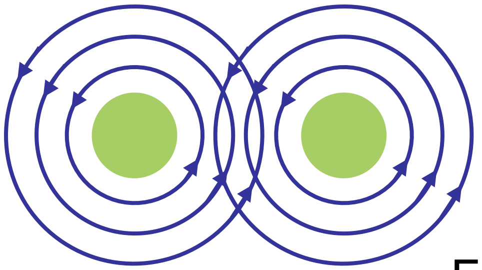
Exact solution: ⇨ $H(0) = \dfrac{\Phi_0}{2\pi\lambda^2} (\ln \kappa - 0.28)$




Energy per unit length of the vortex:

$$\varepsilon = \frac{\Phi_0}{8\pi} H(0) = \left( \frac{\Phi_0}{4\pi\lambda} \right)^2 \ln \kappa$$

# Interaction between vortices



$x = |\vec{r}_1 - \vec{r}_2|$

$$\vec{H} + \lambda^2 \vec{\nabla} \times \vec{\nabla} \times \vec{H} = \Phi_0 [\delta(\vec{r} - \vec{r}_1) + \delta(\vec{r} - \vec{r}_2)]\, \vec{e}_{\mathrm{v}}$$

Free energy $\mathcal{F} = \dfrac{\Phi_0}{8\pi} [H(\vec{r}_1) + H(\vec{r}_2)] = 2\varepsilon + \dfrac{\Phi_0}{8\pi} 2 H_{12}(x)$

$H(\vec{r}_1)$ consists of the field of vortex 1 itself and the field $H_{12}(x)$ due to vortex 2.

Interaction energy: $U(x) = \dfrac{\Phi_0 H_{12}(x)}{4\pi}$ .

Interaction force per unit length: $f = -\dfrac{\mathrm{d}U}{\mathrm{d}x} = -\dfrac{\Phi_0}{4\pi} \dfrac{\mathrm{d}H_{12}}{\mathrm{d}x} = -\dfrac{\Phi_0}{4\pi} \dfrac{4\pi}{c} j_{12}(x)$

**Lorentz force** for a vortex placed at the density of the external current $\vec{j}$ ⇨ $\vec{f}_{\mathrm{L}} = \dfrac{1}{c} [\vec{j} \vec{\Phi}_0]$, where $\vec{\Phi}_0 = \vec{e}_{\mathrm{v}} \Phi_0$