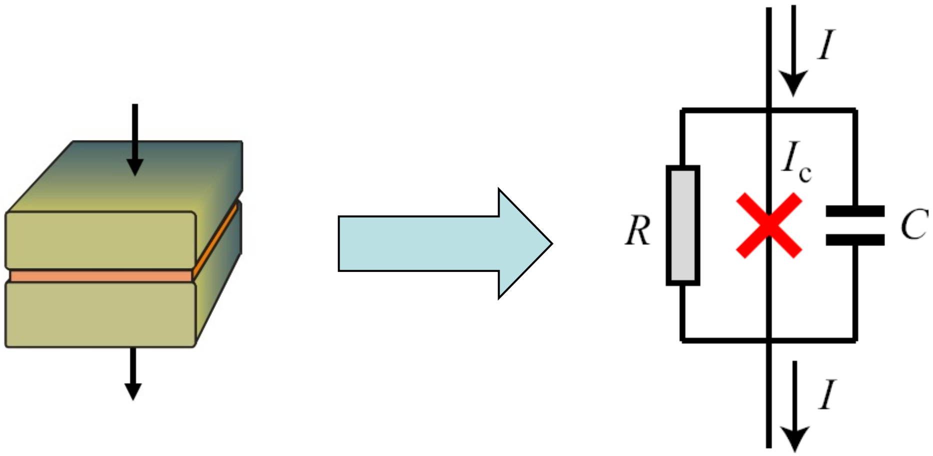
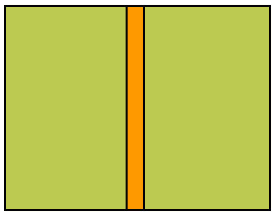
Superconductivity
Lecture 8



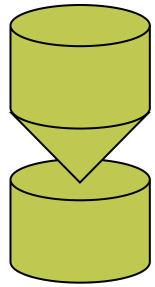
Josephson effect (1)

- Josephson equations
- RCSJ model
- Current-voltage characteristics
- Mechanical analog

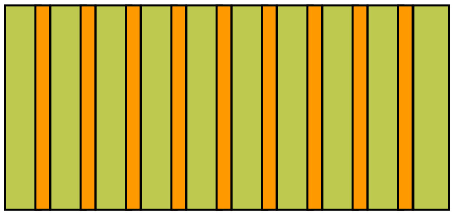
Superconducting weak links



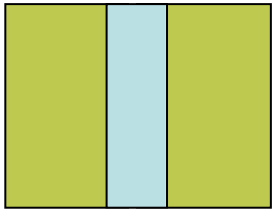
SIS
tunnel junction



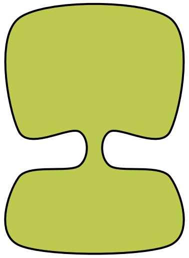
point contact



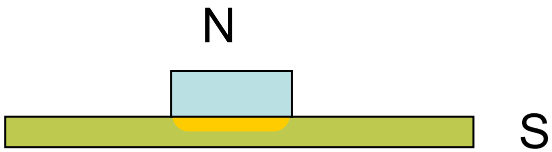
SISISISISISISISISIS
intrinsic junctions
(crystal)



SNS
normal metal

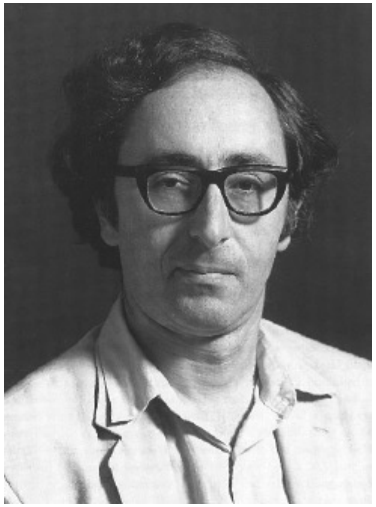


nano-bridge
(constriction)



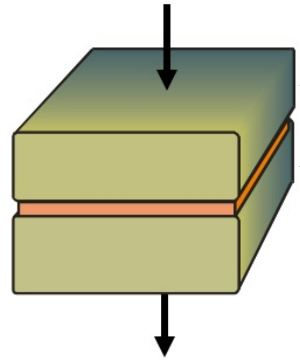
proximity link

Josephson effect



Brian Josephson

Nobel Prize 1973



$$\Psi_1 = |\Psi_1| \exp(i\theta_1)$$

$$\Psi_2 = |\Psi_2| \exp(i\theta_2)$$

$$\varphi = \theta_2 - \theta_1$$

$$I_s(\varphi) = I_c \sin\varphi$$

<u>"Almost self-evident" relations:</u>

(1) If the current through the junction is zero, then $\varphi = 0$

(2) Since a variation of the phase θ of one of the electrodes by does not change anything physically, it is evident that $I_s(\varphi) = I_s(\varphi + 2\pi)$

(3) Changing the sign of the current cause $I_s(\varphi) = -I_s(-\varphi)$

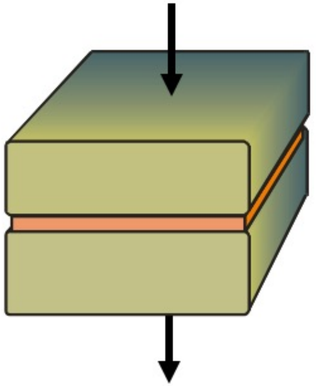
(4) The last relation, $I_s(\pi) = 0$, is somewhat less obvious.

Josephson relations

superconductor

tunnel barrier

superconductor



$$\Psi_1 = |\Psi_1| \exp(i\theta_1)$$

$$\Psi_2 = |\Psi_2| \exp(i\theta_2)$$

$$|\Psi_1| = |\Psi_2|$$

superconducting phase difference: $\varphi = \theta_1 - \theta_2$

Josephson relations

$$\begin{cases} j_s = j_c \sin\varphi \\ V = \dfrac{\hbar}{2e}\dfrac{d\varphi}{dt} \end{cases}$$

Electromagnetic radiation at the frequency $f = \dfrac{V}{\Phi_0}$

$$\Phi_0 = \frac{h}{2e} \approx 2.07 \times 10^{-15}\,\mathrm{V \cdot s}$$

Derivation of Josephson equations (1)

two superconductors forming the Josephson junction satisfy the following system of linearly coupled Schrödinger equations:

$$i\hbar \frac{\partial \Psi_1}{\partial t} = E_1 \Psi_1 + K\Psi_2 \; ; \qquad (1.32)$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = E_2 \Psi_2 + K\Psi_1 \; .$$

Here, E_1 and E_2 are the ground state energies of the superconductors and K is a real coefficient describing the coupling between the two superconductors. When the separation distance between S_1 and S_2 is small enough, the wave functions overlap and $K \neq 0$.

Let us assume that there is a constant potential difference V across the junction. The ground state energies are shifted by

$$E_1 - E_2 = 2eV, \qquad (1.33)$$

where $2e$ is the charge of a pair of electrons. To simplify calculations, the zero of energy can be redefined halfway between the two values E_1 and E_2 such that $E_1 = eV$ and $E_2 = -eV$. Under these conditions

$$i\hbar \frac{\partial \Psi_1}{\partial t} = eV\,\Psi_1 + K\Psi_2 \; ; \qquad (1.34)$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = K\Psi_1 - eV\,\Psi_2 \; . \qquad (1.35)$$

Derivation of Josephson equations (2)

Using $\Psi_1 = |\Psi_1| \exp(\mathrm{i}\,\theta_1)$, $\Psi_2 = |\Psi_2| \exp(\mathrm{i}\,\theta_2)$, and $\varphi = \theta_2 - \theta_1$, and separating the real and imaginary parts we obtain

$$\hbar \frac{\partial(|\Psi_1|^2)}{\partial t} = 2K\,|\Psi_1|\,|\Psi_2| \sin\varphi\,; \tag{1.36}$$

$$\hbar \frac{\partial(|\Psi_2|^2)}{\partial t} = -2K\,|\Psi_1|\,|\Psi_2| \sin\varphi\,; \tag{1.37}$$

$$\hbar \frac{\partial\theta_1}{\partial t} = -K \frac{|\Psi_2|}{|\Psi_1|} \cos\varphi - eV\,; \tag{1.38}$$

$$\hbar \frac{\partial\theta_2}{\partial t} = -K \frac{|\Psi_1|}{|\Psi_2|} \cos\varphi + eV\,. \tag{1.39}$$

Since $|\Psi_1|^2$ and $|\Psi_2|^2$ represent the density of Cooper pairs in each superconductor, the quantities $2e\frac{\partial(|\Psi_1|^2)}{\partial t} = J_1$ and $2e\frac{\partial(|\Psi_2|^2)}{\partial t} = J_2$ are simply the current densities. From Eqs. (1.36), (1.37) we can see that $J_1 = -J_2 = J_\mathrm{s}$.

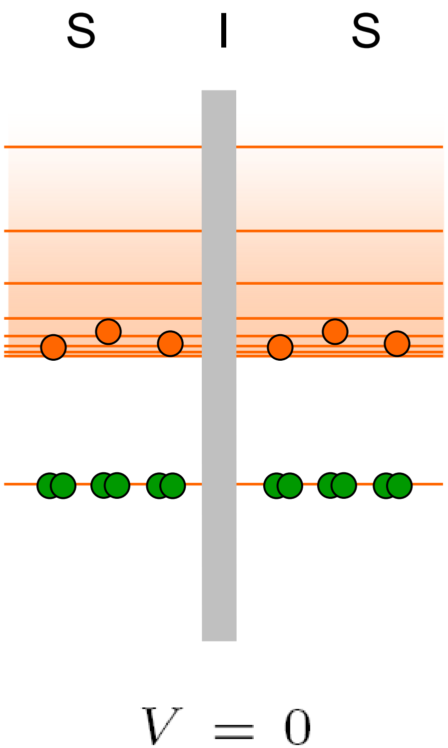
The actual values of $|\Psi_1|^2$ and $|\Psi_2|^2$ are the excess charges supplied by the external source. We then set

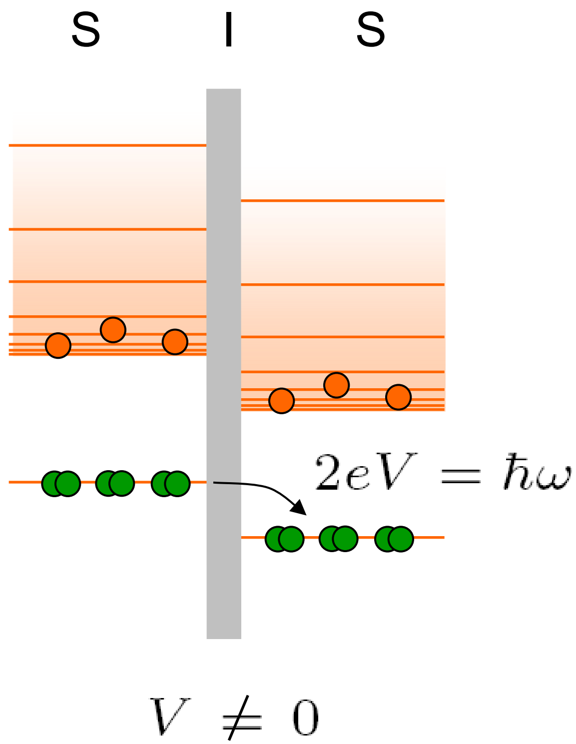
$$|\Psi_1|^2 = |\Psi_2|^2 \approx \rho_0\,; \tag{1.40}$$

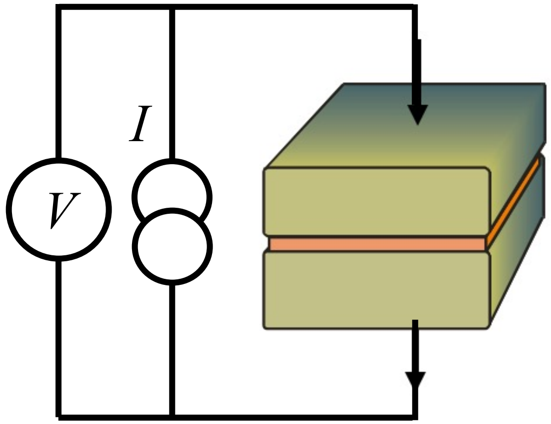
$$2e\frac{2K\rho_0}{\hbar} = J_\mathrm{c}\,. \tag{1.41}$$

Therefore, from (1.36), (1.37) and (1.40), (1.41) ⟹ $I_\mathrm{s}(\varphi) = I_\mathrm{c} \sin\varphi$. Subtraction of (1.38) from (1.39) yields $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{2e}{\hbar} V$

Non-zero voltage state







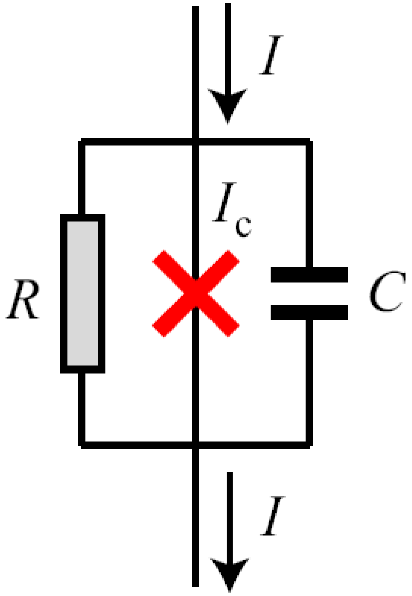
coherent emission
of photons
with frequency

$$f = \frac{V}{\Phi_0}$$

$$\Phi_0 = \frac{h}{2e} \approx 2.07 \times 10^{-15}\,\mathrm{V \cdot s}$$

RSCJ model

RSCJ = Resistively-capasitively shunted junction



$$I = I_c \sin\varphi + \frac{V}{R} + C\frac{\mathrm{d}V}{\mathrm{d}t}$$

Since $\frac{\mathrm{d}\varphi}{\mathrm{d}t} \equiv \dot{\varphi} = \frac{2\pi}{\Phi_0}V$, we get

$$I = I_c \sin\varphi + \frac{\Phi_0}{2\pi R}\dot{\varphi} + \frac{\Phi_0 C}{2\pi}\ddot{\varphi}.$$

In dimensionless units $i \equiv \frac{I}{I_c}$ and $\tau = t\left(\frac{2\pi I_c}{\Phi_0 C}\right)^{1/2}$

we have finally $$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\tau^2} + \beta_c^{-1/2}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + \sin\varphi = i,$$

where $\beta_c = 2\pi I_c R^2 C/\Phi_0$ is the McCumber parameter.

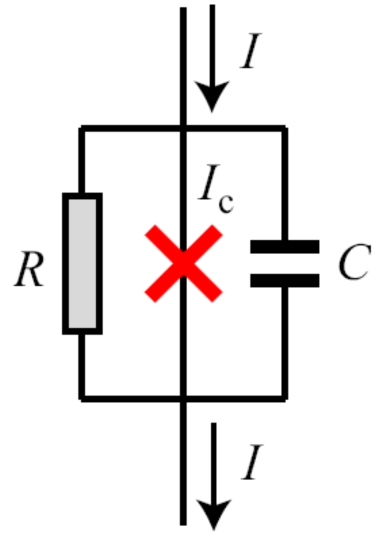
Josephson plasma frequency

The dimensionless time $\tau = \omega_{\mathrm{p}} t$ is defined through the *Josephson plasma frequency*

$$f_{\mathrm{p}} = \frac{\omega_{\mathrm{p}}}{2\pi} = \left(\frac{I_{\mathrm{c}}}{2\pi\,\Phi_0 C} \right)^{1/2}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\tau^2} + \beta_{\mathrm{c}}^{-1/2} \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + \sin\varphi = i,$$

This frequency determines the characteristic time scale of the dynamical processes in the junction.



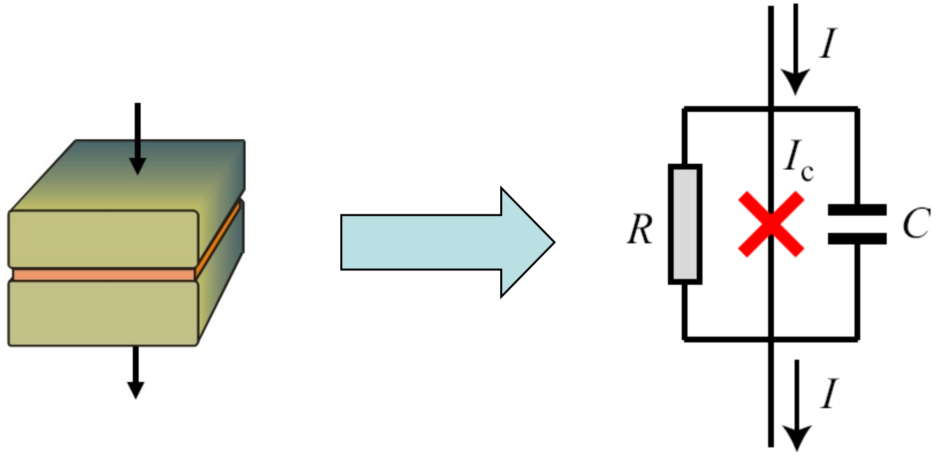
A Josephson tunnel junction can be viewed as a nonlinear oscillator placed in a lossy medium.

The characteristic "quality factor" of the oscillator is $Q = \sqrt{\beta_{\mathrm{c}}} = \omega RC$

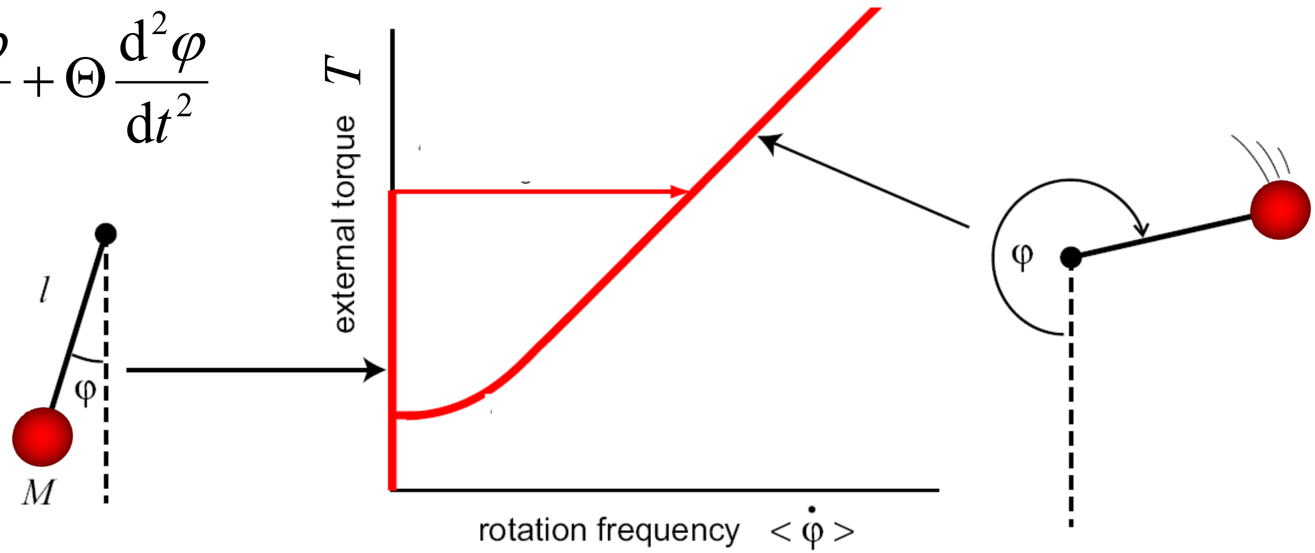
Mechanical analog of a Josephson junction: driven underdamped pendulum

$$I = I_C \sin\varphi + \frac{V}{R} + C\frac{\mathrm{d}V}{\mathrm{d}t}$$

where $$V = \frac{\hbar}{2e}\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

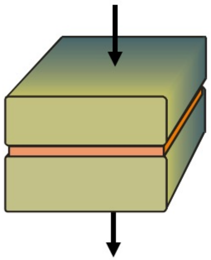


$$T = Mgl\sin\varphi + \zeta\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \Theta\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}$$

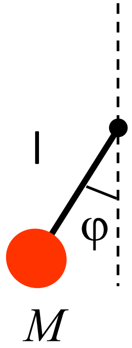


Mapping between Josephson junction and mechanical pendulum

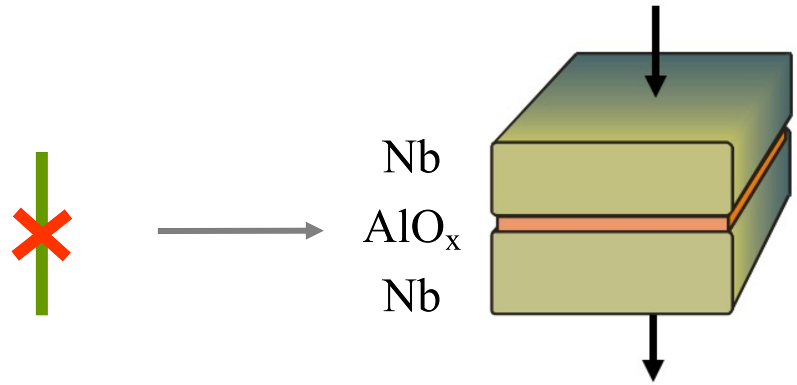
$$I = I_C \sin\varphi + \frac{1}{R}\frac{\hbar}{2e}\frac{d\varphi}{dt} + C\frac{\hbar}{2e}\frac{d^2\varphi}{dt^2} \qquad T = Mgl\sin\varphi + \zeta\frac{d\varphi}{dt} + \Theta\frac{d^2\varphi}{dt^2}$$

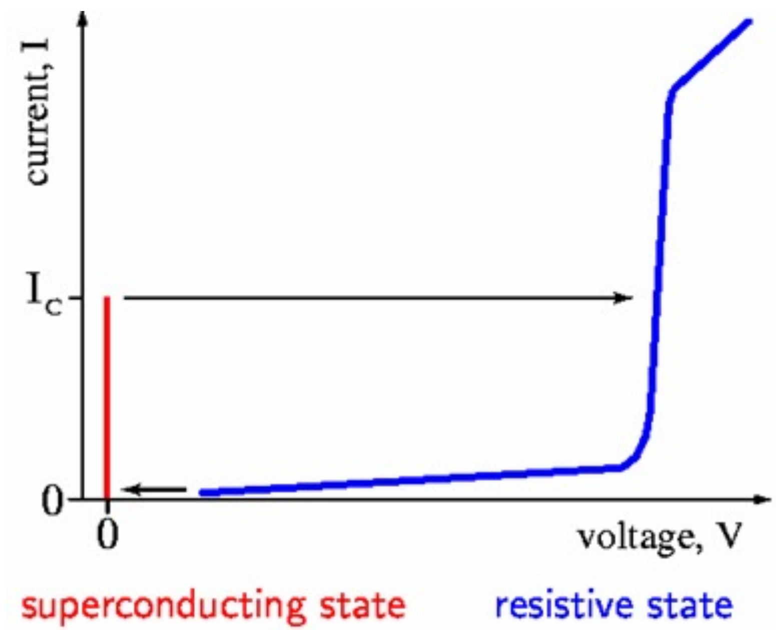


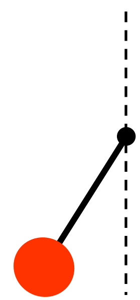
Josephson junction	mechanical pendulum
phase difference φ	angle from vertical φ
voltage $V = \Phi_0\dot{\varphi}/(2\pi)$	angular velocity $\dot{\varphi}$
critical current I_c	restoring constant Mgl
conductance R^{-1}	damping coefficient ζ
capacitance C	moment of inertia Ml^2
bias current I	external torque $\mathcal{T}$
Josephson plasma frequency f_p	oscillation frequency $f_0 = \sqrt{g/l}/(2\pi)$



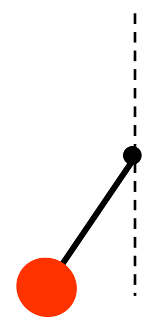
Current-voltage characteristics





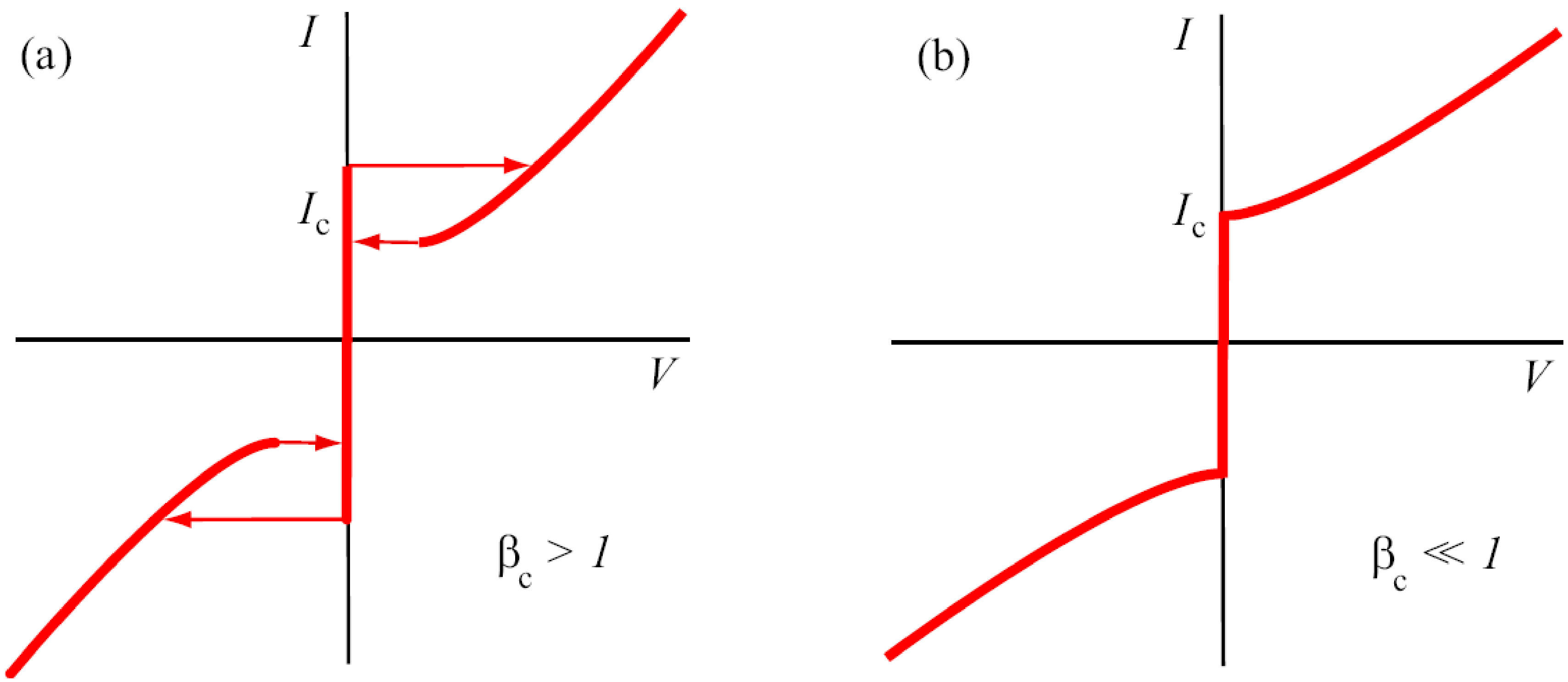


superconducting state



resistive state

Effect of damping

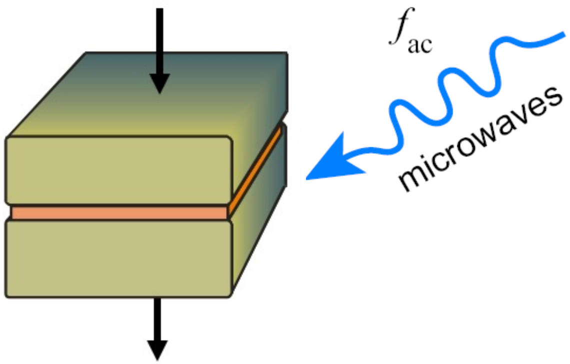


underdamped
(large R and C)

overdamped
(small R and C)

$$\beta_c = 2\pi\, I_c R^2 C/\Phi_0$$

Irradiation my microwaves



Voltage bias: $V + V_{ac}\cos(2\pi f_{ac} t)$

Integrating the 2nd Josephson equation:

$$\varphi = \varphi_0 + \frac{2\pi}{\Phi_0} V t + \frac{V_{ac}}{\Phi_0 f_{ac}} \sin(2\pi f_{ac} t)$$

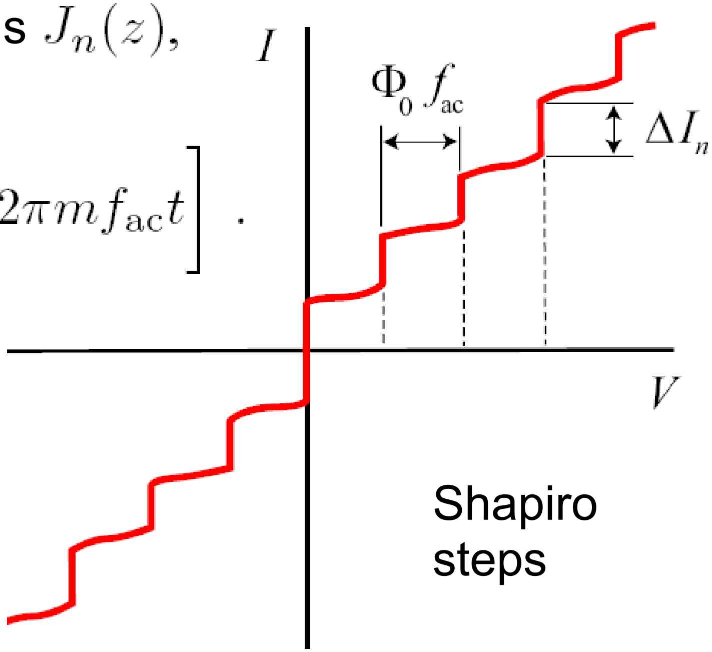
Expressing $\sin(z \sin x)$ in terms of Bessel functions $J_n(z)$, we get

$$I_s = I_c \sum_{m=0}^{\infty} (-1)^m J_m \left(\frac{V_{ac}}{\Phi_0 f_{ac}}\right) \sin\left[\varphi_0 + \frac{2\pi}{\Phi_0} V t - 2\pi m f_{ac} t\right] .$$

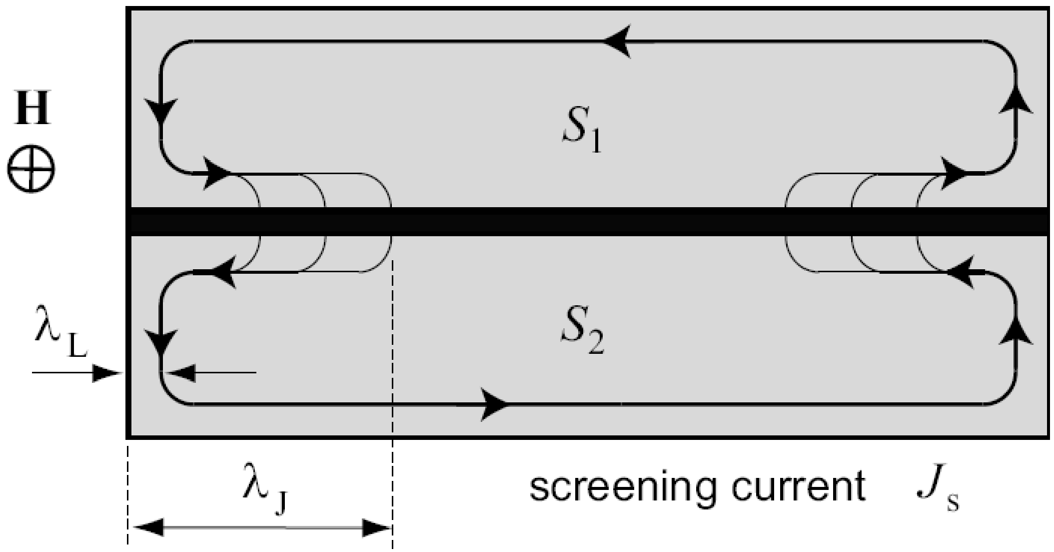
A time-independent contribution to the current occurs at

$$V = V_m = m f_{ac} \Phi_0 , \qquad m = 0, \pm 1, \pm 2, \ldots$$

Current steps: $\Delta I_m \simeq I_c J_m \left(\frac{V_{ac}}{\Phi_0 f_{ac}}\right) .$



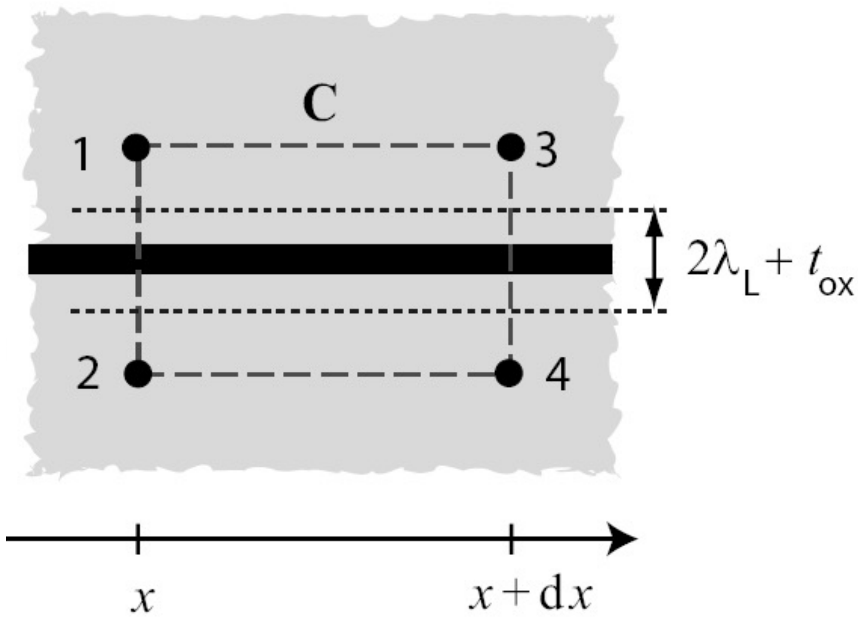
Effect of magnetic field



The size of the region penetrated by the current and the magnetic field is

$$\Lambda = 2\lambda_{\rm L} + t_{\rm ox}$$

Integration contour inside a Josephson junction



Remind the canonical momentum of a Cooper pair $\mathbf{p} \equiv \hbar \nabla \theta = 2m\, \mathbf{v}_s + 2e\, \mathbf{A}\,.$

Let us perform integration along the contour $\mathbf{C}$. We obtain

$$\hbar \int_1^3 \nabla\theta \cdot \mathrm{d}\mathbf{l} + \hbar \int_4^2 \nabla\theta \cdot \mathrm{d}\mathbf{l} = 2e \int_1^3 \mathbf{A} \cdot \mathrm{d}\mathbf{l} + 2e \int_4^2 \mathbf{A} \cdot \mathrm{d}\mathbf{l}\,.$$

Ferrell-Prange equation

$$\hbar(\theta_3 - \theta_1 + \theta_2 - \theta_4) = 2e \oint_{\mathbf{C}} \mathbf{A} \cdot \mathrm{d}\mathbf{l} \quad \Rightarrow \quad \hbar\left[(\theta_3 - \theta_4) - (\theta_1 - \theta_2)\right] = 2e\,\mathrm{d}\Phi$$

where $\mathrm{d}\Phi$ is the magnetic flux enclosed in $\mathbf{C}$.

$$\hbar\left[\varphi(x + \mathrm{d}x) - \varphi(x)\right] = 2e\mu_0 \Lambda H(x)\,\mathrm{d}x$$, where $\Lambda = 2\lambda_\mathrm{L} + t_\mathrm{ox}$

$$\Rightarrow \quad H(x) = \frac{\Phi_0}{2\pi\mu_0\Lambda}\frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

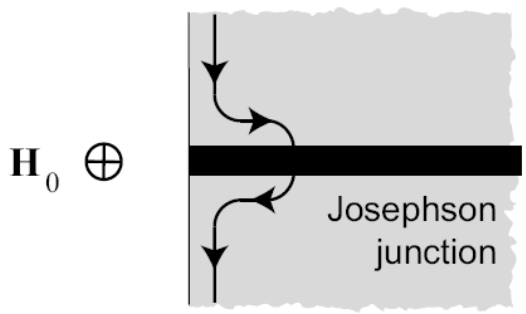
According to the Maxwell's equation $j_\mathrm{s} = \frac{\mathrm{d}H}{\mathrm{d}x}$ and Josephson equation $j_\mathrm{s} = j_\mathrm{c}\sin\varphi$

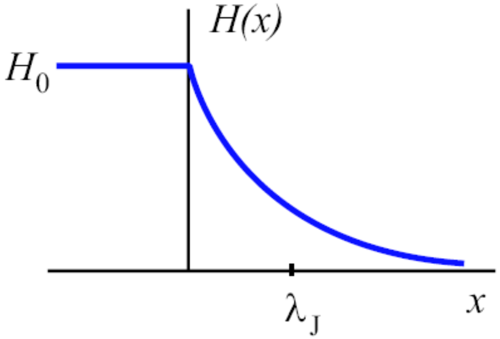
$$\Rightarrow \quad \boxed{\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{1}{\lambda_\mathrm{J}^2}\sin\varphi}$$

where $\lambda_\mathrm{J} = \sqrt{\frac{\Phi_0}{2\pi\mu_0\Lambda j_\mathrm{c}}}$ is the Josephson length.

(Josephson penetration depth)

Long junction in weak magnetic field





$$\frac{d^2\varphi}{dx^2} = \frac{1}{\lambda_J^2}\sin\varphi$$

Let us assume first $|\varphi| \ll 1$

$$\frac{d^2\varphi}{dx^2} \approx \frac{1}{\lambda_J^2}\varphi$$

$$\varphi(x) = \varphi(0)\exp\left(-\frac{x}{\lambda_J}\right)$$

$$H(x) = \frac{\Phi_0}{2\pi\mu_0\Lambda}\frac{d\varphi}{dx} = H_0\exp\left(-\frac{x}{\lambda_J}\right)$$