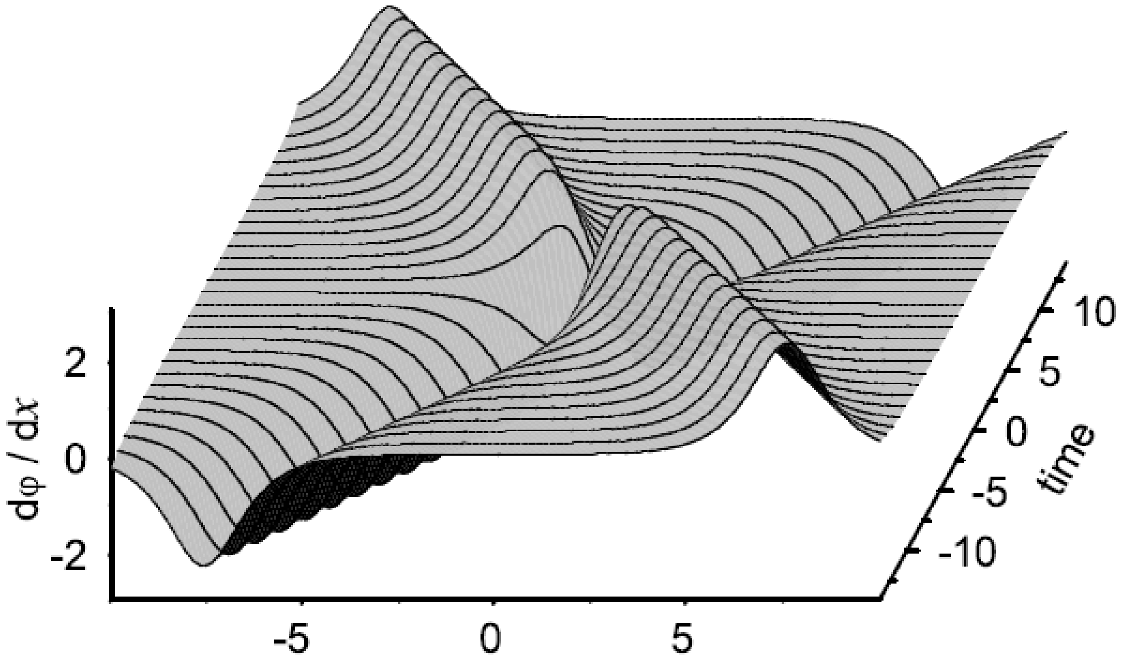
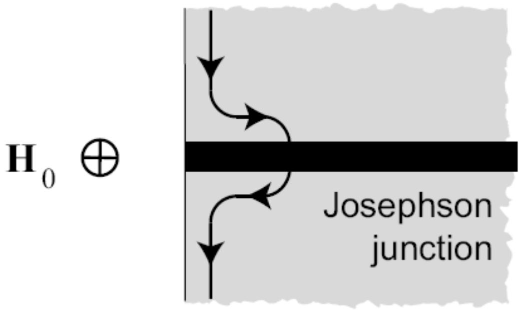
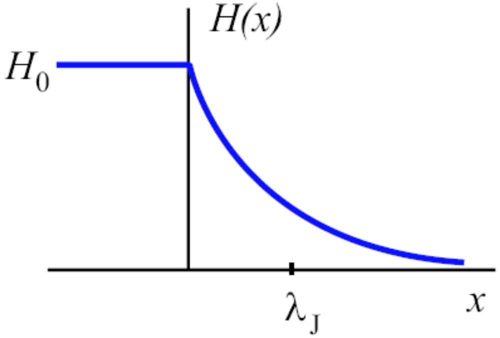
Superconductivity
Lecture 9



Long junction in weak magnetic field





Ferrell-Prange equation:

$$\frac{d^2\varphi}{dx^2} = \frac{1}{\lambda_J^2} \sin\varphi$$

Let us assume first $|\varphi| \ll 1$

$$\frac{d^2\varphi}{dx^2} \approx \frac{1}{\lambda_J^2} \varphi$$

$$\varphi(x) = \varphi(0) \exp\left(-\frac{x}{\lambda_J}\right)$$

$$H(x) = \frac{\Phi_0}{2\pi\mu_0\Lambda} \frac{d\varphi}{dx} = H_0 \exp\left(-\frac{x}{\lambda_J}\right)$$

Short junction in magnetic field

Consider a short junction $L \ll \lambda_J$

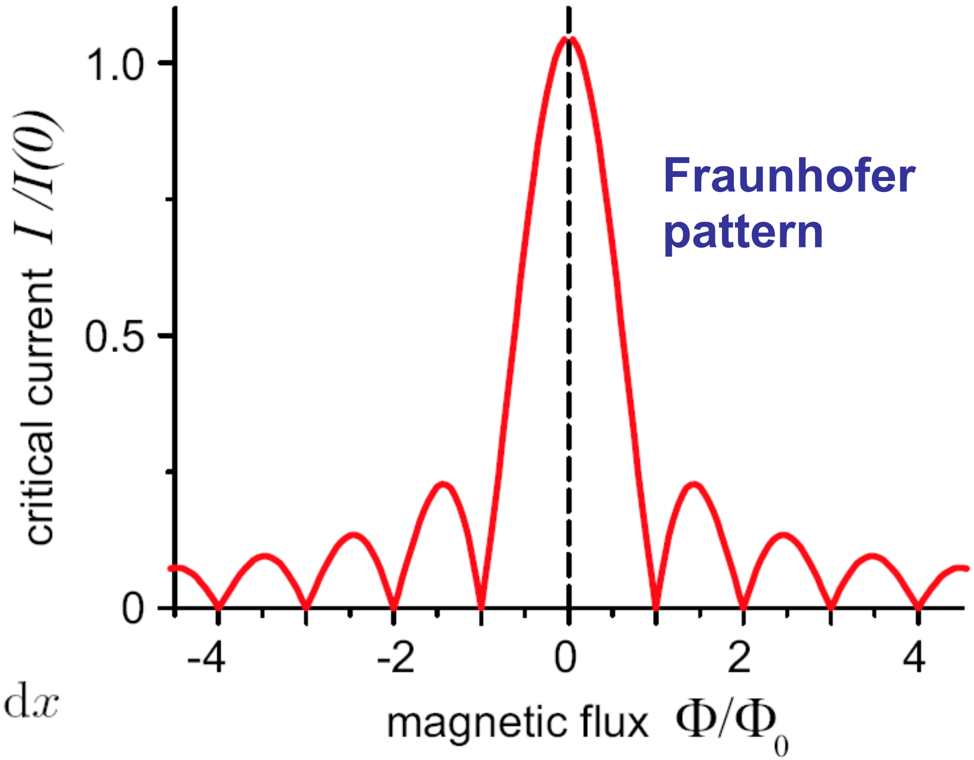
$$\varphi(x) = \frac{2\pi\mu_0\Lambda}{\Phi_0} H_0 x + C_0$$

$$j_s(x) = j_c \sin\left[\frac{2\pi\mu_0\Lambda}{\Phi_0} H_0 x + C_0\right]$$

$$I_s = j_c \int_{-W/2}^{W/2} \mathrm{d}y \int_{-L/2}^{L/2} \sin\left(\frac{2\pi}{\phi} x + C_0\right) \mathrm{d}x$$

$$= j_c L W \frac{\phi}{\pi L} \sin\frac{\pi L}{\phi} \sin C_0 \,,$$

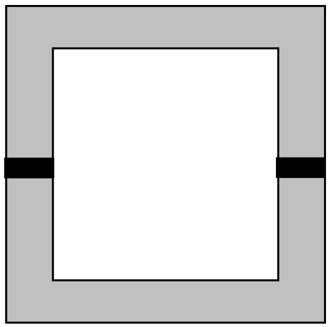
where $\phi = \Phi_0/(\mu_0 \Lambda H_0)$



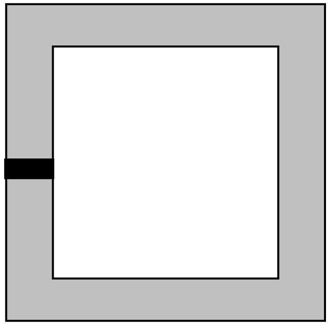
$$I_s^{\max} \equiv I_c(H_0) = I_c(0) \frac{\sin(\pi\Phi/\Phi_0)}{(\pi\Phi/\Phi_0)}$$

where $\Phi = \mu_0 \Lambda H_0 L$

Superconducting quantum interferometers



dc SQUID
(2 junctions)



rf SQUID
(1 junction)

Superconducting QUantum Interference Devices (SQUIDs) have opened new horizons in measurement techniques.

SQUID-based instruments are unique in their sensitivity.

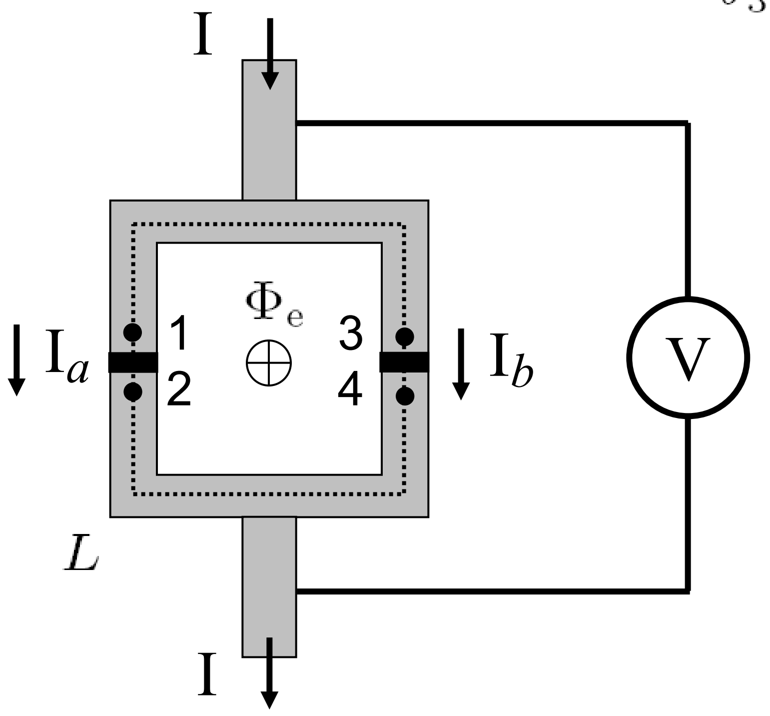
SQUID magnetometers are able to resolve flux increments of ~ 10^{-10} G.

Precision SQUID voltmeters reach the sensitivity of ~ 10^{-15} V.

Two-junction (dc) SQUID

dc = "direct current"

two-junction SQUID



$$\theta_3 - \theta_1 + \theta_2 - \theta_4 = \frac{2e}{\hbar}\left(\int_1^3 \vec{A}\,\mathrm{d}\vec{l} + \int_4^2 \vec{A}\,\mathrm{d}\vec{l}\right)$$

The term $2m\vec{v}_\mathrm{s}$ is omitted because the contour passes everywhere through the interior of the superconductor, well away from the edges.

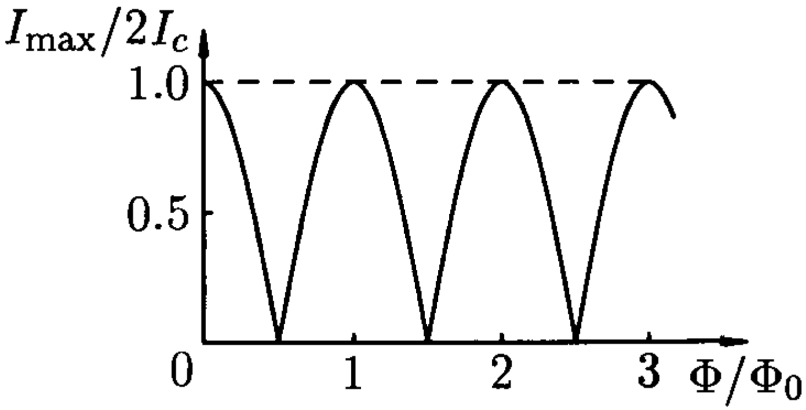
$$\varphi_a = \theta_2 - \theta_1, \;\; \varphi_b = \theta_4 - \theta_3$$

$$\varphi_a - \varphi_b = \frac{2e}{\hbar}\oint \vec{A}\,\mathrm{d}\vec{l} = 2\pi\Phi/\Phi_0$$

$$I = I_\mathrm{c}\left(\sin\varphi_\mathrm{a} + \sin\varphi_\mathrm{b}\right) = 2I_\mathrm{c}\,\cos\frac{\pi\Phi}{\Phi_0}\,\sin\left(\varphi_\mathrm{b} + \frac{\pi\Phi}{\Phi_0}\right)$$

Magnetic field pattern of dc SQUID

$$I_{\mathrm{max}} = 2I_{\mathrm{c}} \left|\cos\left(\pi\Phi/\Phi_0\right)\right|$$

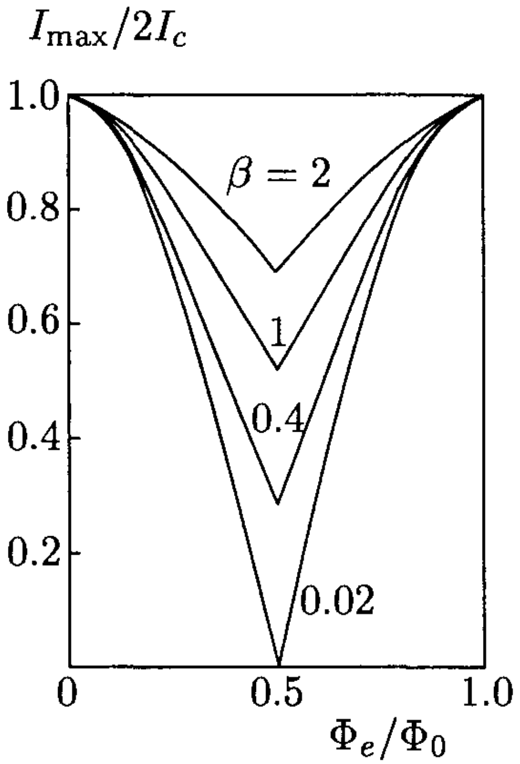


Dependence of the maximum supercurrent through the two-junction interferometer on the total magnetic flux through its interior.

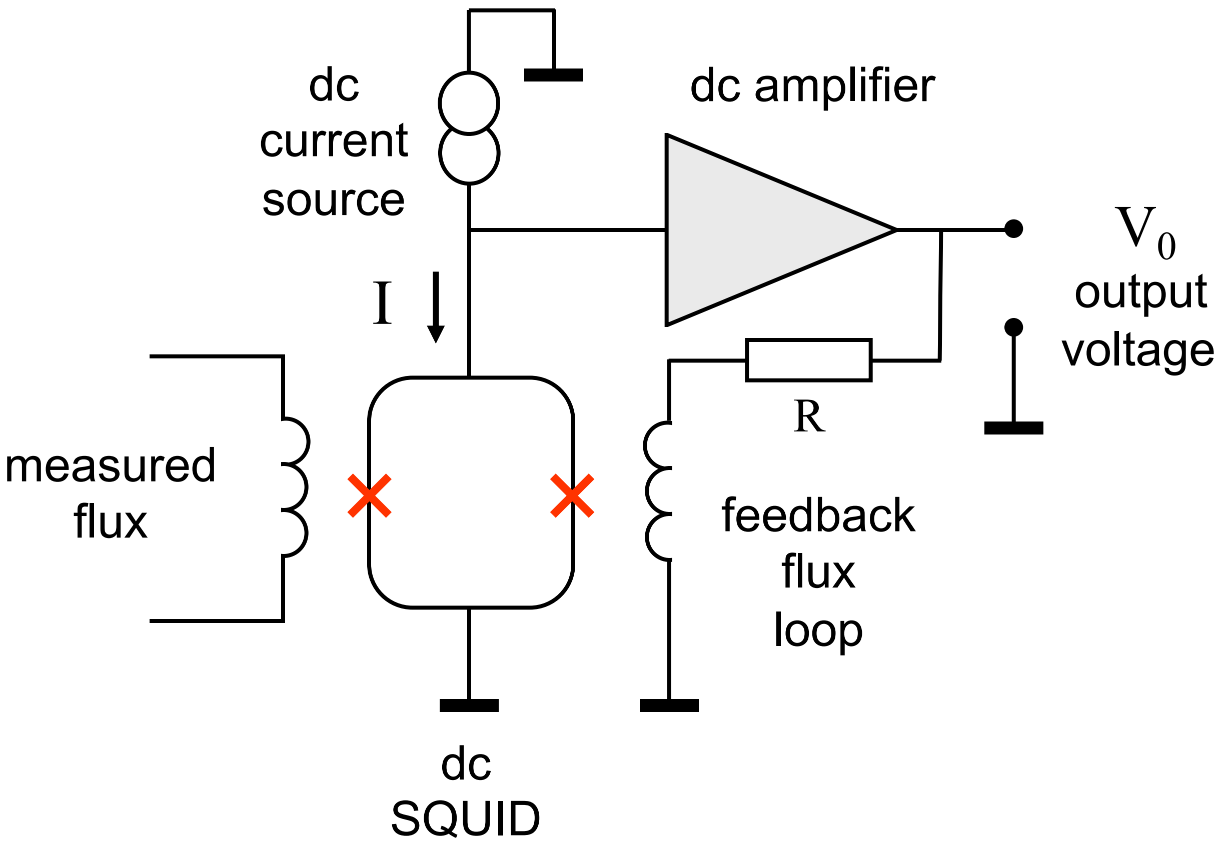
$$\beta = 2LI_{\mathrm{c}}/\Phi_0$$

$$\Phi = \Phi_{\mathrm{e}} - LI_{\mathrm{sc}}$$

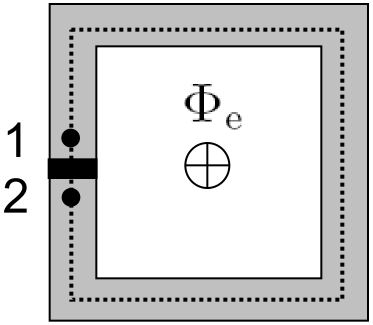
I_{sc} is the screening current



dc SQUID operation principle



Single junction (rf) SQUID



L

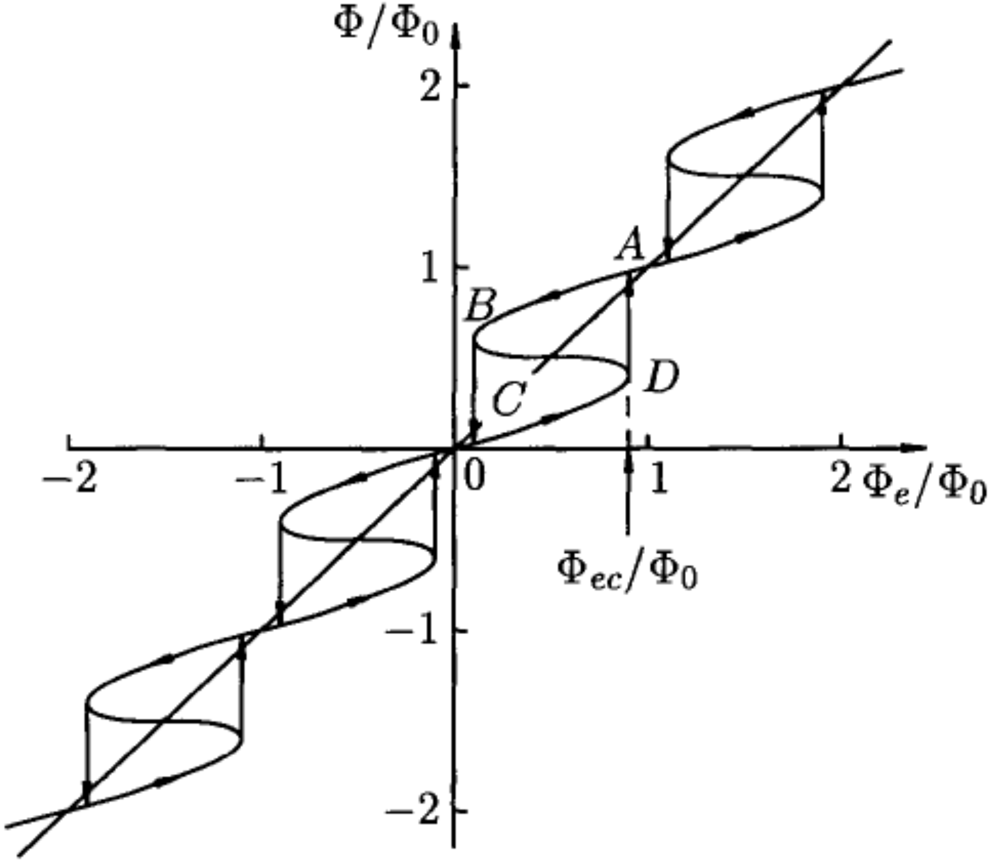
loop inductance

$$\hbar\,(\theta_2 - \theta_1) = 2e \int_1^2 \vec{A}\,\mathrm{d}\vec{l}$$

$$\hbar\varphi = 2e \oint \vec{A}\,\mathrm{d}\vec{l}$$

$$\varphi = 2\pi\Phi/\Phi_0$$

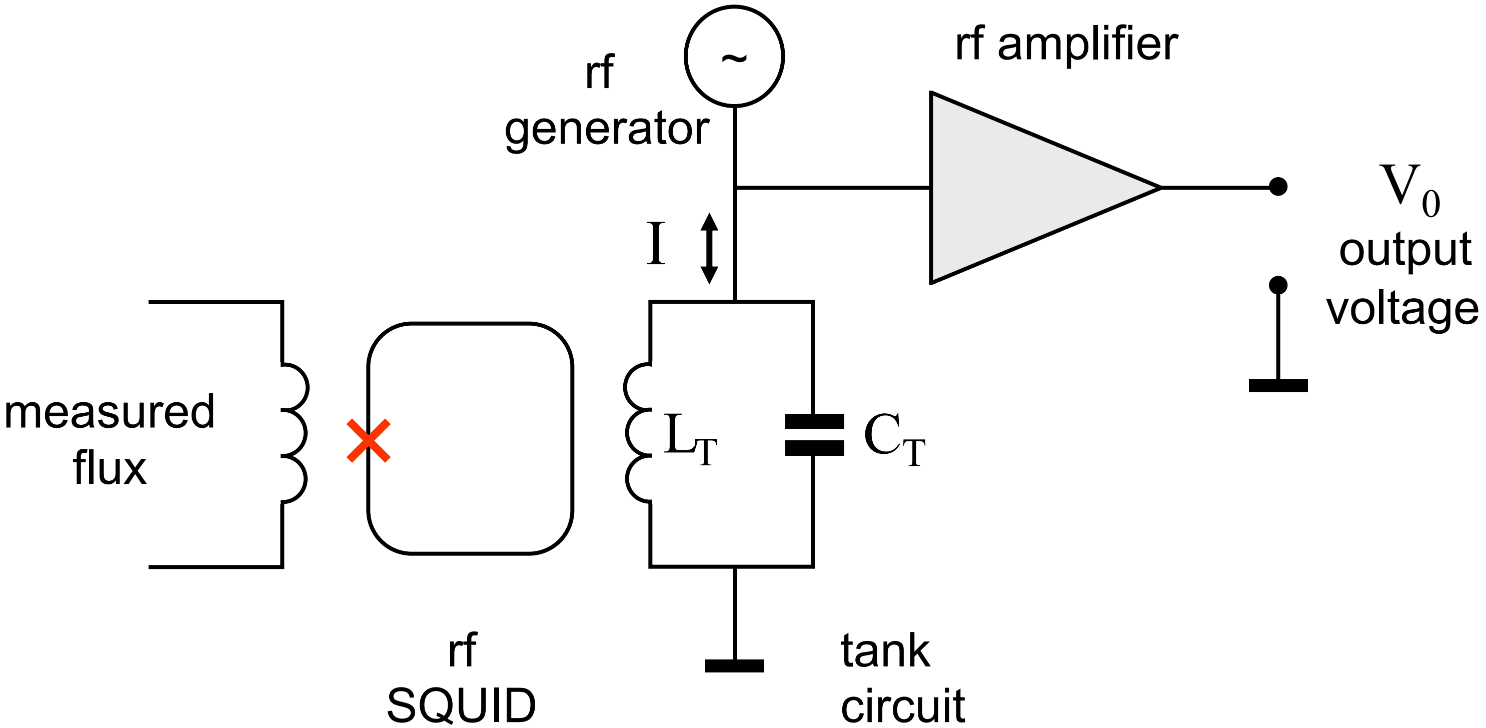
$$\Phi = \Phi_e - L I_{sc}$$



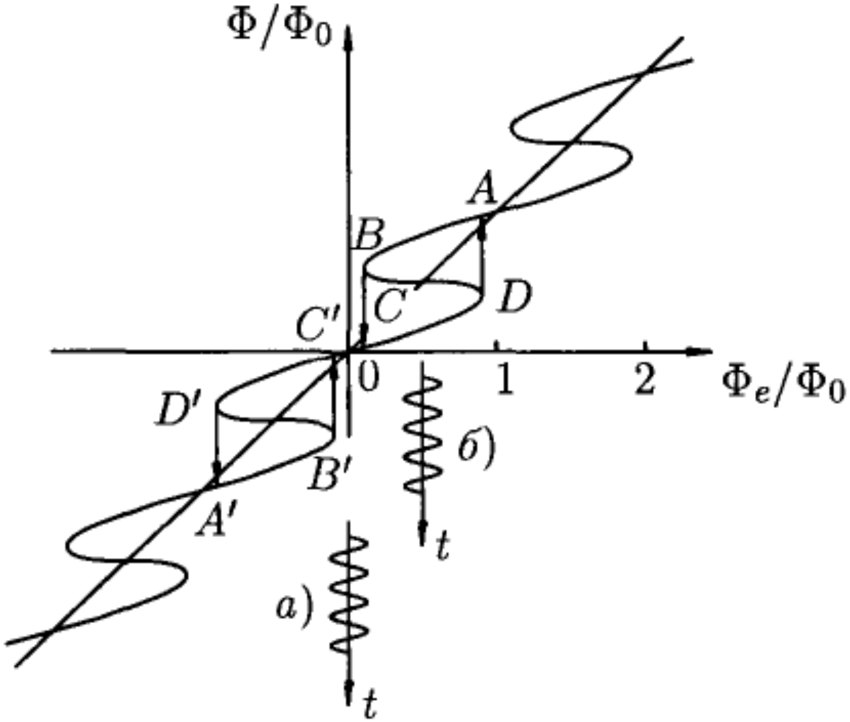
we get implicit relation between Φ and Φ_e :

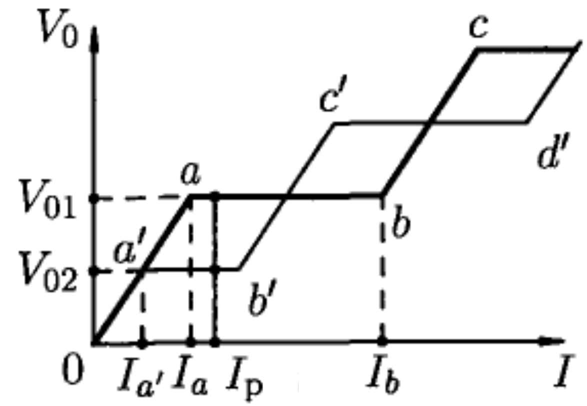
$$\Phi_e = \Phi + L I_c \sin(2\pi\Phi/\Phi_0)$$

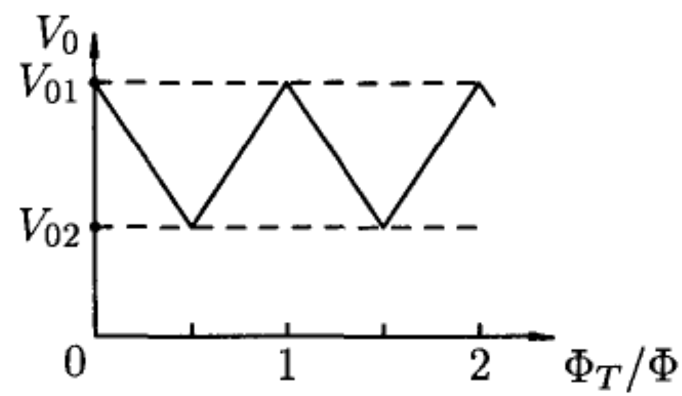
rf SQUID operation principle



rf SQUID characteristics



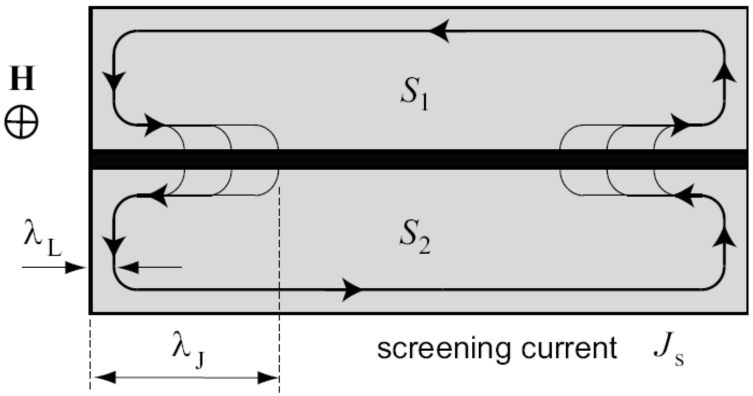


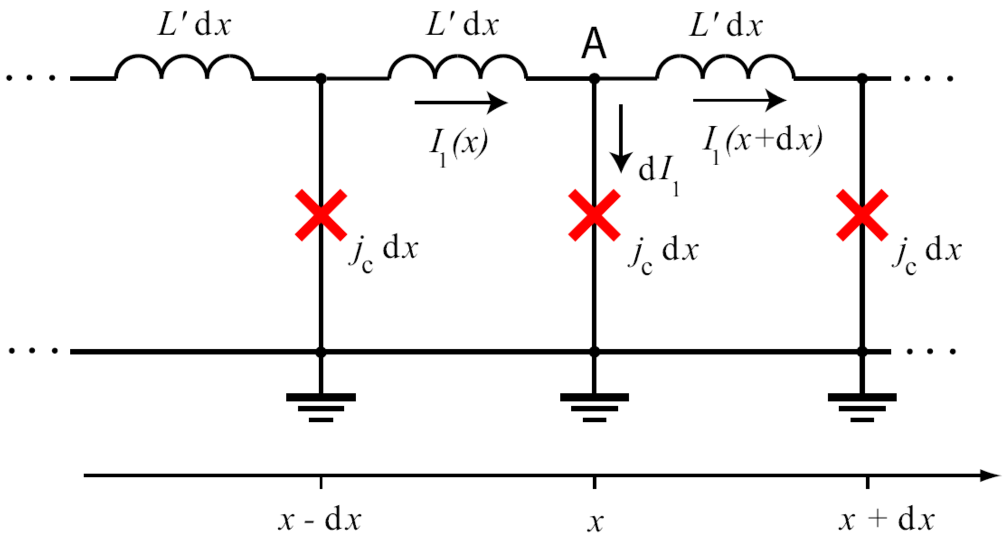


Long Josephson junctions

- Long junction in magnetic field
- Time-dependent dynamics
- Perturbed sine-Gordon equation
- Plasma waves
- Solitons and antisolitons, breathers
- Multi-soliton solutions
- Junction geometries
- Perturbations and power balance

Ferrell-Prange equation



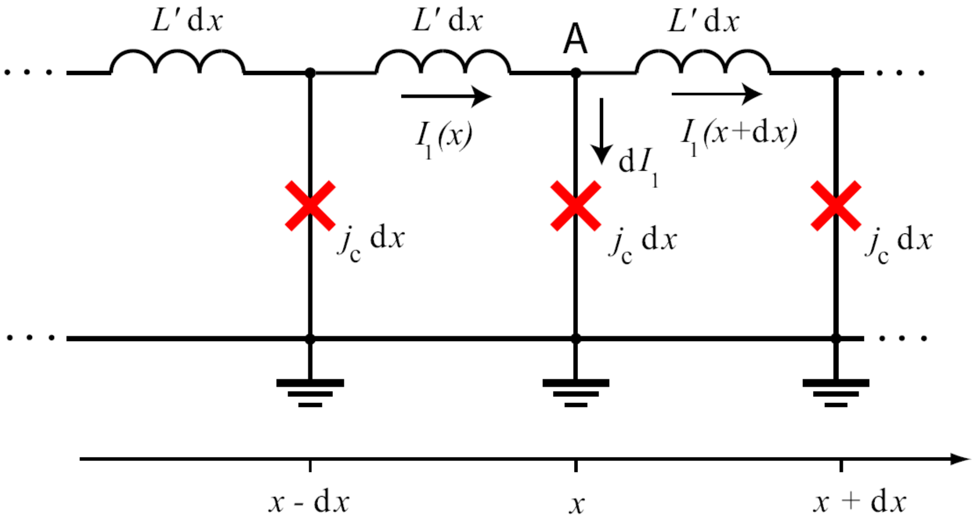


Ferrell-Prange equation:

(1) $$\frac{d^2\varphi}{dx^2} = \frac{1}{\lambda_J^2}\sin\varphi$$

where $\lambda_J = \sqrt{\dfrac{\Phi_0}{2\pi\mu_0\Lambda j_c}}$ is the Josephson penetration depth

Equivalent circuit



$$I_1(x) = \frac{\varphi(x) - \varphi(x - \mathrm{d}x)}{L'\,\mathrm{d}x}\frac{\Phi_0}{2\pi}\,;$$

$$I_1(x + \mathrm{d}x) = \frac{\varphi(x + \mathrm{d}x) - \varphi(x)}{L'\,\mathrm{d}x}\frac{\Phi_0}{2\pi}\,;$$

$$dI_1 = j_\mathrm{c}\,\mathrm{d}x\,\sin\varphi\,.$$

The Kirchhoff current law for point A yields

$$I_1(x + \mathrm{d}x) - I_1(x) = dI_1$$

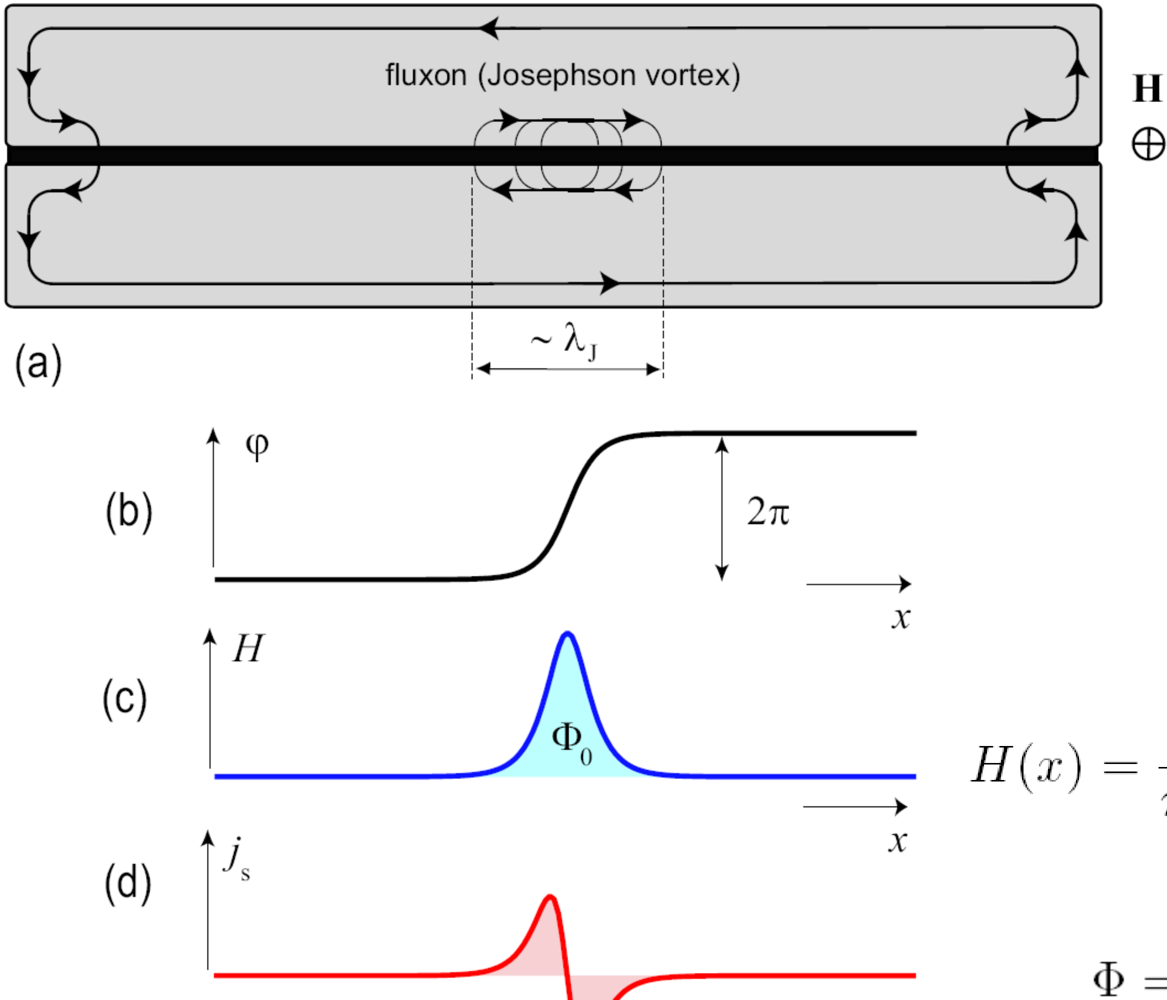
$$\frac{\mathrm{d}I_1}{\mathrm{d}x} = \frac{\Phi_0}{2\pi}\frac{\varphi(x + \mathrm{d}x) - 2\varphi(x) + \varphi(x - \mathrm{d}x)}{L'\,\mathrm{d}x^2} = j_\mathrm{c}\sin\varphi$$

⇨ Ferrell-Prange equation:

$$\frac{\Phi_0}{2\pi L'}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = j_\mathrm{c}\sin\varphi\,,$$

$$L' = \Phi_0/(2\pi j_\mathrm{c}\lambda_\mathrm{J}^2) = \mu_0\Lambda$$

Fluxon in long junction



$$j_s(x) = j_c \sin\varphi(x)$$

$$\frac{d^2\varphi}{dx^2} = \frac{1}{\lambda_J^2}\sin\varphi$$

there is an exact solution

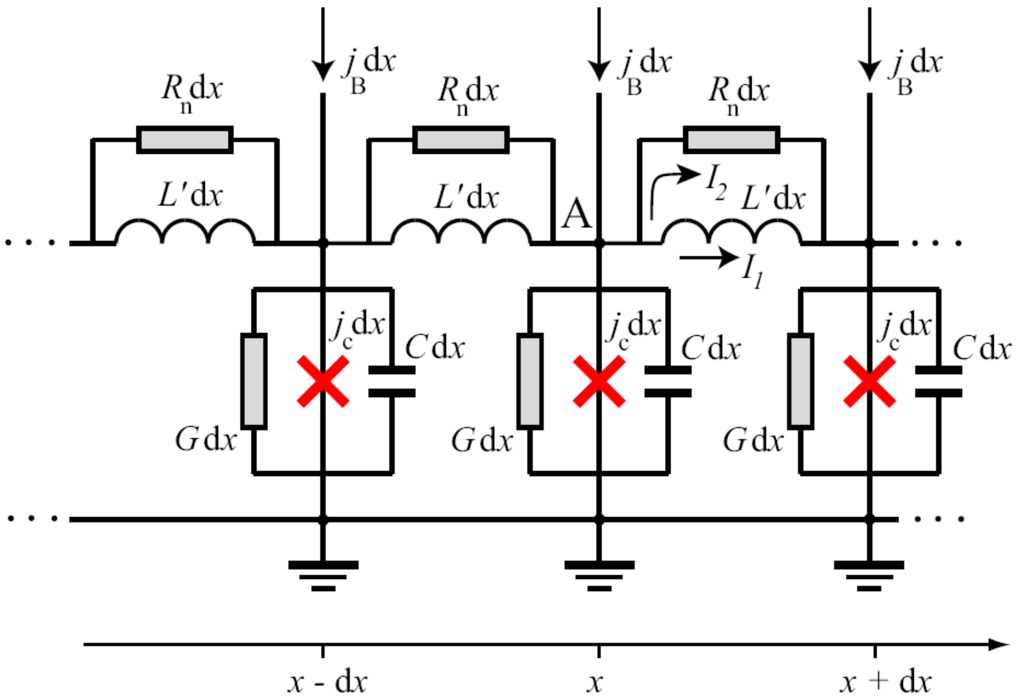
$$\varphi(x) = 4\arctan\left[\exp\frac{x-x_0}{\lambda_J}\right]$$

corresponding to a fluxon

$$H(x) = \frac{\Phi_0}{\pi\mu_0\Lambda\lambda_J}\frac{2\exp\left[(x-x_0)/\lambda_J\right]}{1+\exp\left[2(x-x_0)/\lambda_J\right]}$$

$$\Phi = \frac{\Phi_0}{2\pi}\int_{-\infty}^{+\infty}\frac{d\varphi(x)}{dx}dx = \Phi_0$$

Time-dependent equivalent circuit



Kirchhoff current law for the node A:

$$\frac{\partial(I_1 + I_2)}{\partial x} = j_B - C\frac{\partial V}{\partial t} - \frac{V}{R} - j_c \sin\varphi$$

The voltage drop in the *x*-direction

$$dV = -R_n I_2\, dx = -L' \frac{\partial I_1}{\partial t}\, dx$$

from the 2nd Josephson relation

$$\frac{\partial I_1}{\partial t} = -\frac{1}{L'}\frac{\partial V}{\partial x} = -\frac{\Phi_0}{2\pi L'}\frac{\partial^2 \varphi}{\partial x\, \partial t}$$

$$\Rightarrow \quad I_1 = -\frac{\Phi_0}{2\pi L'}\frac{\partial \varphi}{\partial x}\, ;$$

$$I_2 = -\frac{1}{R_n}\frac{\partial V}{\partial x} = -\frac{\Phi_0}{2\pi R_n}\frac{\partial^2 \varphi}{\partial x\, \partial t}\, .$$

Phase dynamics in a long junction

Inserting I_1 and I_2 into $\frac{\partial(I_1+I_2)}{\partial x} = j_\mathrm{B} - C\frac{\partial V}{\partial t} - \frac{V}{R} - j_\mathrm{c}\sin\varphi$ we get

$$\frac{\partial^2\varphi}{\partial x^2} - L'C\frac{\partial^2\varphi}{\partial t^2} = \frac{2\pi L'}{\Phi_0} j_\mathrm{c}\left(\sin\varphi - \frac{j_\mathrm{B}}{j_\mathrm{c}}\right) + \frac{L'}{R}\frac{\partial\varphi}{\partial t} - \frac{L'}{R_\mathrm{n}}\frac{\partial^3\varphi}{\partial x^2 \partial t} \qquad (2)$$

electric and magnetic fields $E = \frac{V}{t_\mathrm{ox}} = \frac{1}{t_\mathrm{ox}}\frac{\Phi_0}{2\pi}\frac{\mathrm{d}\varphi}{\mathrm{d}t}$; $H = \frac{1}{\mu_0\Lambda}\frac{\Phi_0}{2\pi}\frac{\mathrm{d}\varphi}{\mathrm{d}x}$.

The angular Josephson plasma frequency $\omega_\mathrm{p} = \sqrt{\frac{2\pi j_\mathrm{c}}{\Phi_0 C}}$

The specific inductance and capacitance per unit area of junction $\begin{cases} L' = \mu_0\Lambda \\ C = \frac{\epsilon\epsilon_0}{t_\mathrm{ox}} \end{cases}$

The velocity of the propagation of electromagnetic waves in a long Josephson junction $\bar{c} = \lambda_\mathrm{J}\,\omega_\mathrm{p} = \frac{1}{\sqrt{L'C}} = c\sqrt{\frac{t_\mathrm{ox}}{\epsilon\Lambda}}$ is called the *Swihart velocity*

Perturbed sine-Gordon equation

Using normalized units $\frac{x}{\lambda_J} \to x$ and $\frac{t}{\omega_p^{-1}} \to t$ we can rewrite Eq.(2) as

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi + \alpha\varphi_t - \beta\varphi_{xxt} - \gamma \qquad (3)$$

Here we have introduced a compact notation for the derivatives:

$$\varphi_{xx} \equiv \frac{\partial^2\varphi}{\partial x^2}\,; \quad \varphi_{tt} \equiv \frac{\partial^2\varphi}{\partial t^2}\,; \quad \varphi_t \equiv \frac{\partial\varphi}{\partial t}\,; \quad \varphi_{xxt} \equiv \frac{\partial^3\varphi}{\partial x^2\partial t}\,.$$

The last three dimensionless coefficients in Eq.(3) are defined as

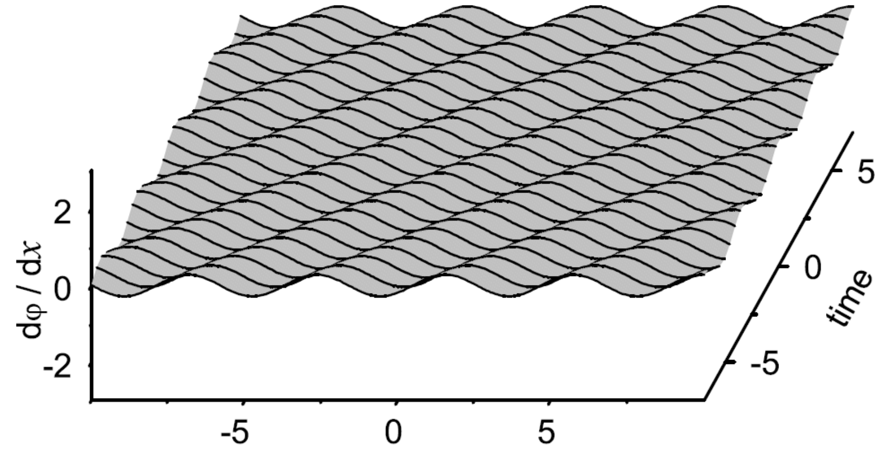
$$\underbrace{\alpha \equiv \sqrt{\frac{\Phi_0}{2\pi j_c R^2 C}} = \frac{1}{RC\omega_p}}_{\text{tunneling of quasiparticles}}\,, \quad \underbrace{\beta \equiv \sqrt{\frac{2\pi j_c (L')^2}{\Phi_0 C R_n^2}} = \frac{\omega_p L'}{R_n}}_{\text{surface currents damping}}\,, \quad \underbrace{\gamma \equiv \frac{j_B}{j_c}}_{\text{bias current}}$$

Josephson plasma waves

Let us first assume $\alpha = \beta = \gamma = 0$

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi \quad (3)$$

(unperturbed) sine-Gordon equation



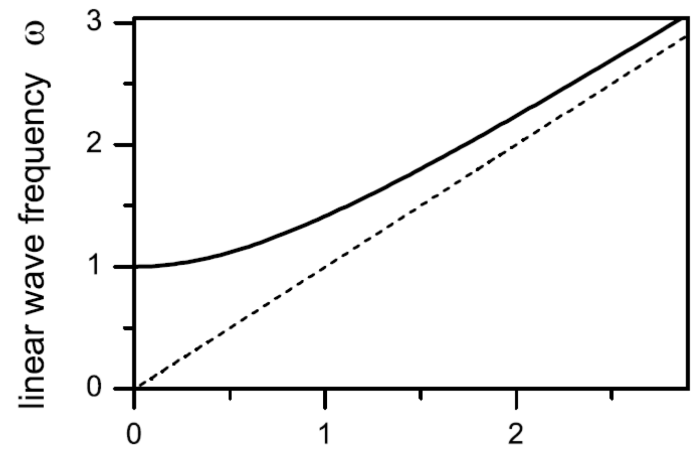
Small-amplitude waves $|\varphi| \ll 1$

$$\varphi_{xx} - \varphi_{tt} = \varphi$$

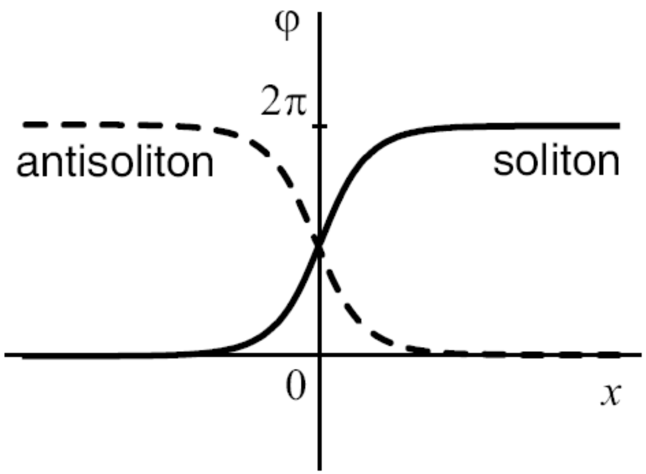
$$\varphi(x,t) = \varphi_0 \exp\left[\mathrm{i}(kx - \omega t)\right] \ ,$$

dispersion relation

$$\omega = \sqrt{1 + k^2} \ .$$



Solitons and antisolitons

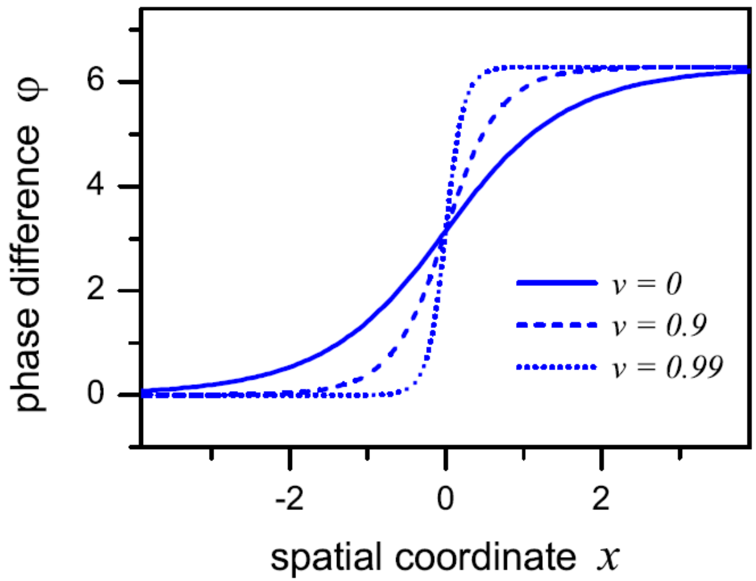


soliton

$$\varphi(x,t) = \varphi_{\mathrm{F}} \equiv 4\arctan\left[\exp\left(\frac{x - vt - x_0}{\sqrt{1-v^2}}\right)\right]$$

antisoliton

$$\varphi(x,t) = \varphi_{\bar{\mathrm{F}}} \equiv 4\arctan\left[\exp\left(-\frac{x - vt - x_0}{\sqrt{1-v^2}}\right)\right]$$



These solutions are invariant with respect to the *Lorentz transformation*:

$$x \quad \Rightarrow \quad x' = \frac{x - vt}{\sqrt{1-v^2}}\,;$$

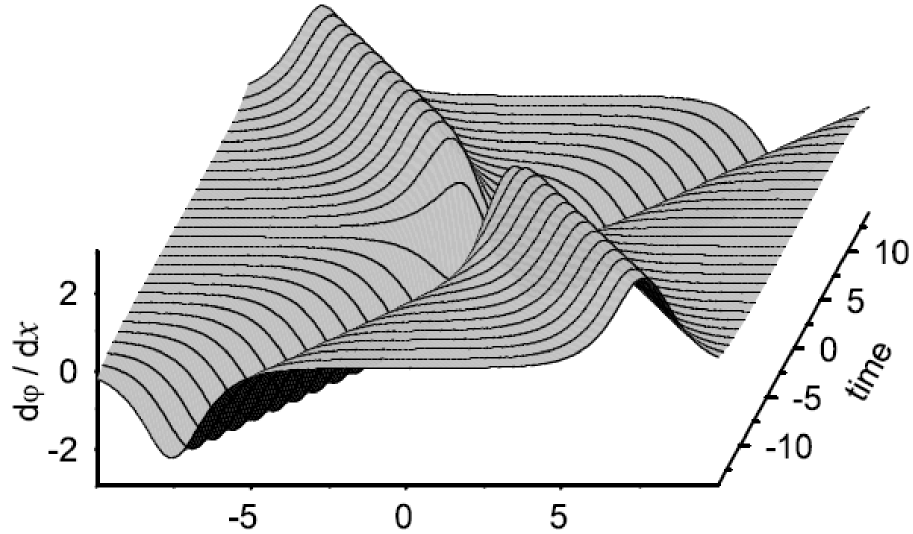
$$t \quad \Rightarrow \quad t' = \frac{t - x/v}{\sqrt{1-v^2}}\,.$$

velocity-dependent “mass” $$m(v) = \frac{8}{\sqrt{1-v^2}}\,.$$

Multi-soliton solutions

A collision between a soliton and an antisoliton is described by the solution:

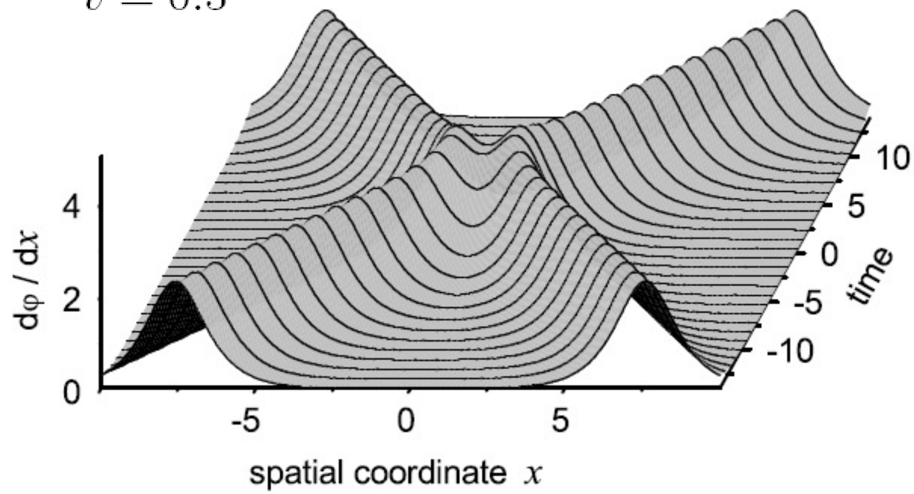
$$\varphi_{F\bar{F}} = 4\arctan\left[\frac{1}{v}\frac{\sinh\left(\frac{vt}{\sqrt{1-v^2}}\right)}{\cosh\left(\frac{x}{\sqrt{1-v^2}}\right)}\right]$$



$v = 0.5$

Solution for a soliton-soliton collision:

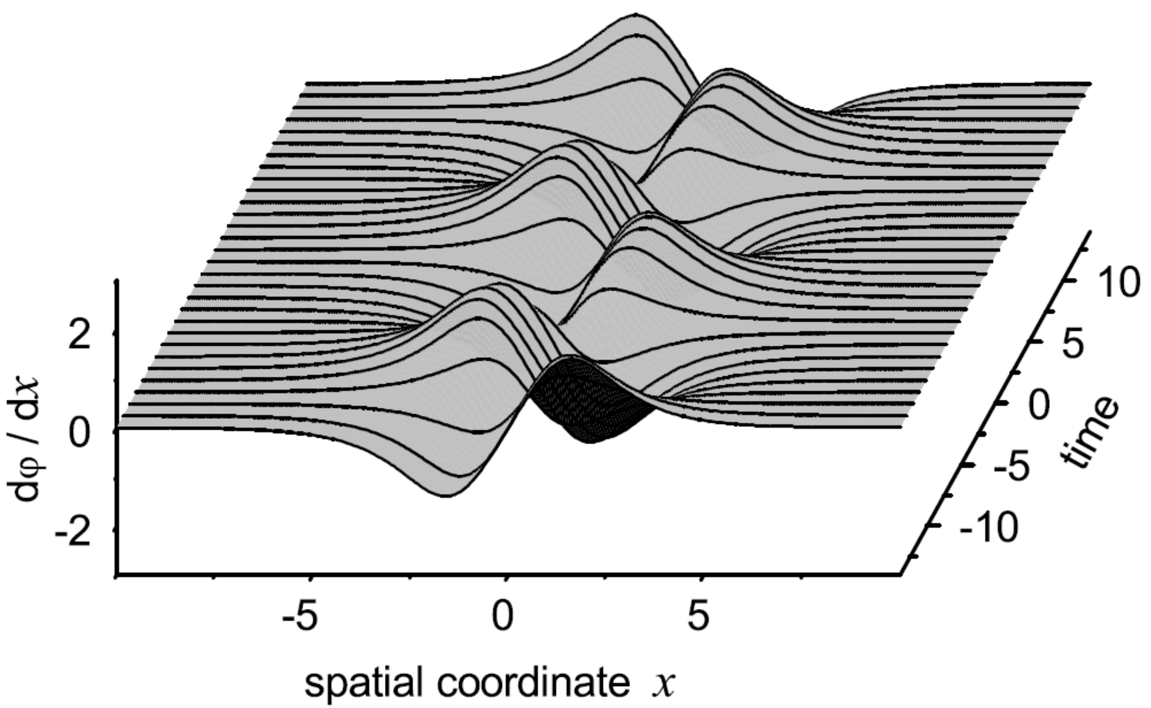
$$\varphi_{\mathrm{FF}} = 4\arctan\left[v\frac{\sinh\left(\frac{x}{\sqrt{1-v^2}}\right)}{\cosh\left(\frac{vt}{\sqrt{1-v^2}}\right)}\right]$$



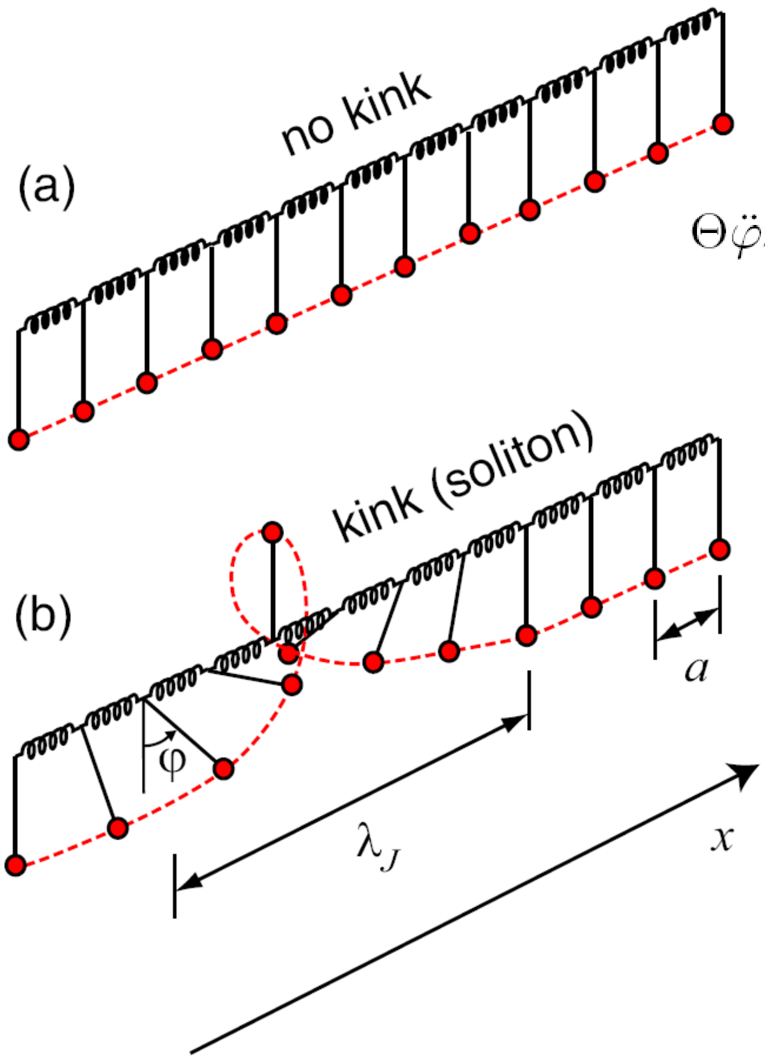
Breather

A bound state, in which the soliton and antisoliton oscillate around their common center of mass, is called a *breather*.

$$\varphi(x,t) = 4\arctan\left[\tan\theta \frac{\sin(t\cos\theta)}{\cosh(x\sin\theta)}\right], \quad \text{where } 0<\theta<\pi/2.$$



Mechanical analog of a long junction



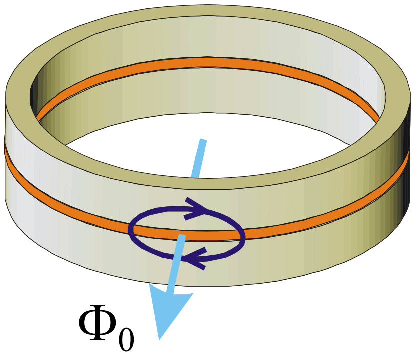
A chain of coupled pendulums

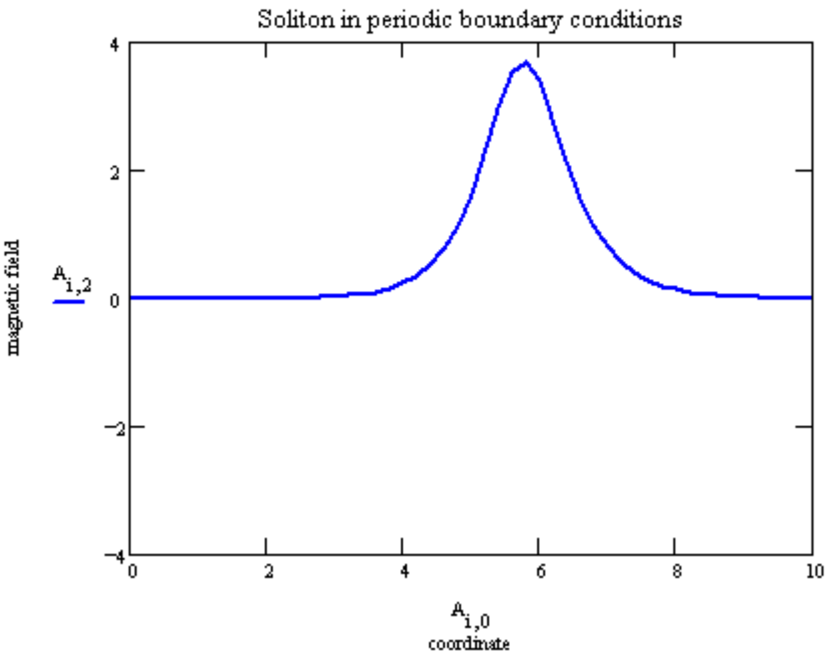
$$\Theta\ddot{\varphi}_i - \varrho(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}) + \zeta\dot{\varphi}_i + Mgl\sin\varphi_i = T$$

$$\alpha = \frac{\zeta}{\sqrt{\Theta M g l}}, \quad \gamma = \frac{T}{Mgl}$$

symbol	electrical quantity	mechanical quantity
φ	phase difference	pendulum angle
φ_x	magnetic field	angle variation along chain
φ_t	voltage	angular velocity
ϱ	$(\text{inductance})^{-1}$	elastic coupling constant
Θ	capacitance	momentum of inertia
Mgl	critical current	gravitational torque
T	bias current	external torque
ζ	conductance	damping coefficient

A soliton in annular Josephson junction





Periodic boundary conditions

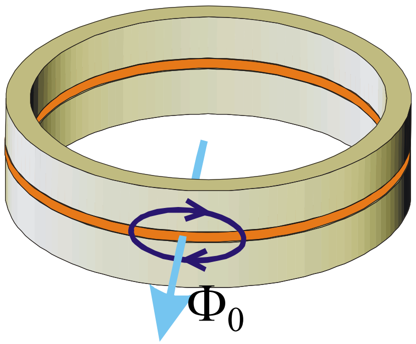
$$\varphi(0) = \varphi(L) + 2\pi n$$

$$\varphi_x(0) = \varphi_x(L)$$

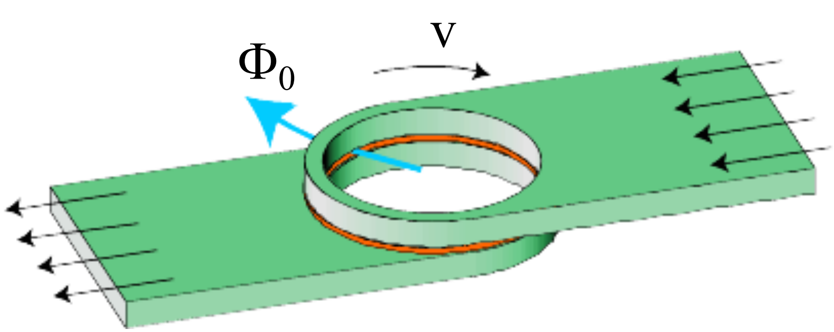
number of trapped flux quanta

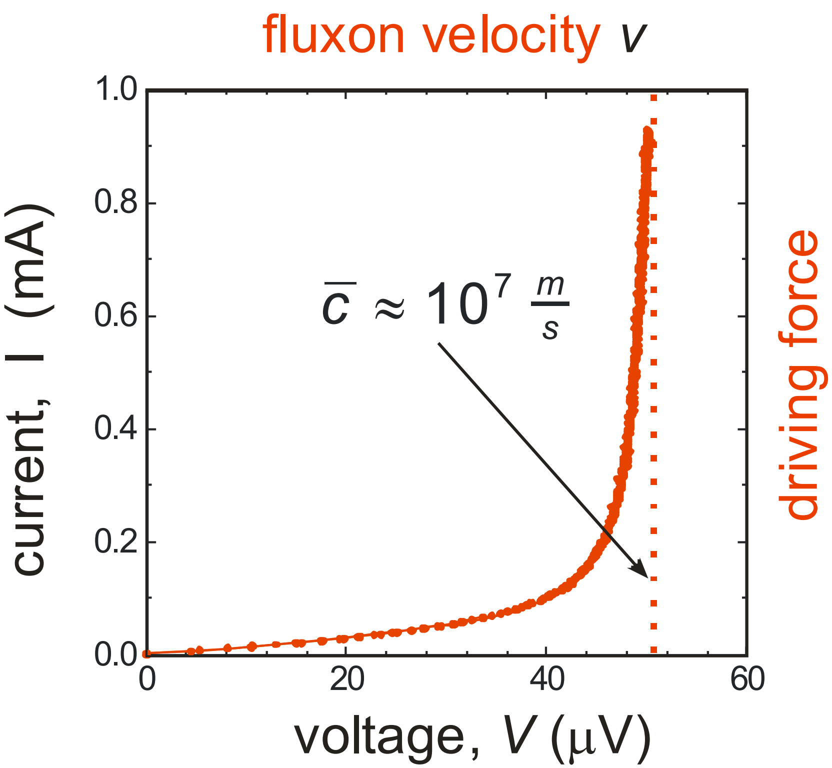
$n = 1$

Measurement of a soliton trapped in a ring



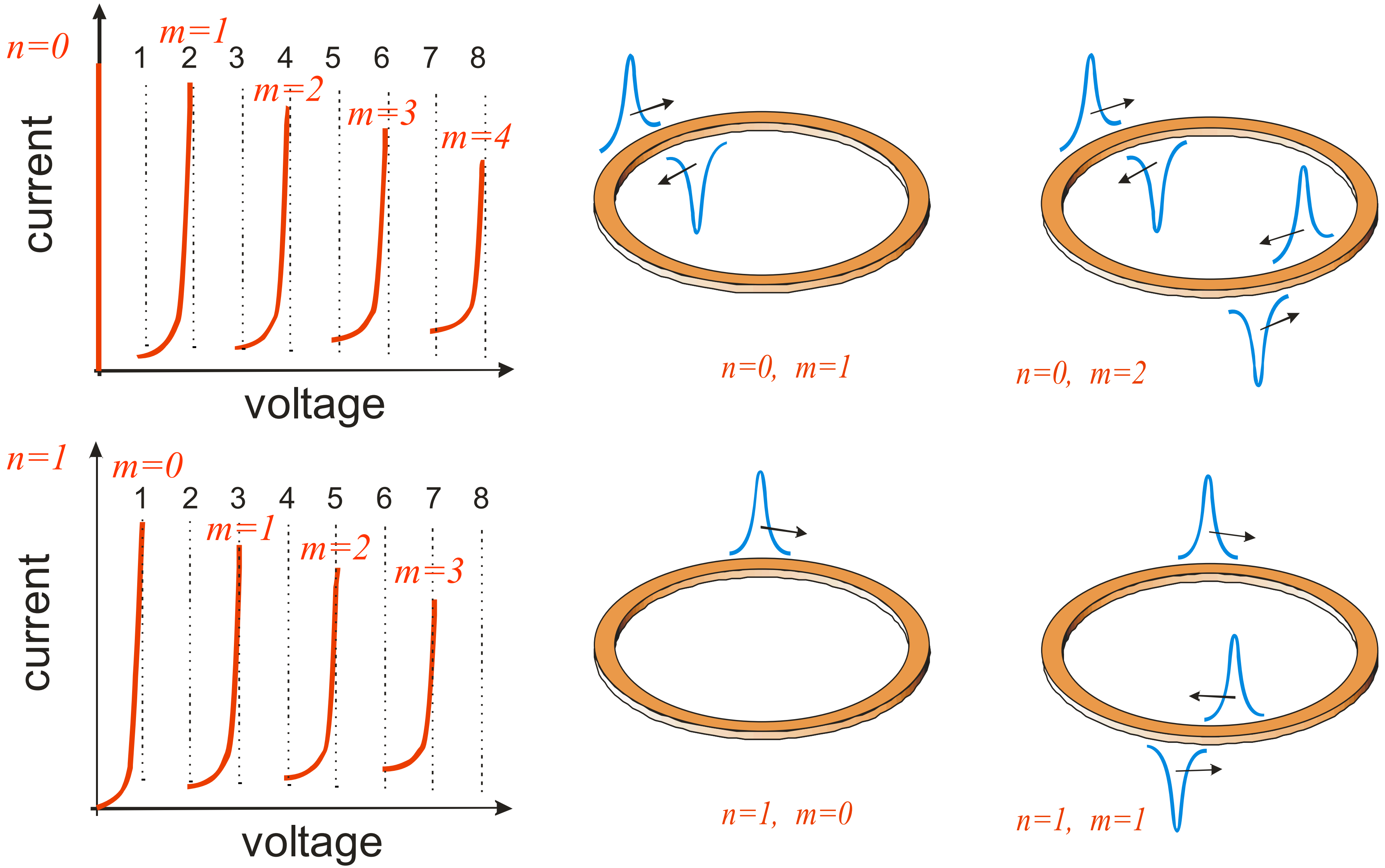
vortex velocity $$v = 2\pi R \frac{V}{\Phi_0}$$





Measurement of a single Josephson vortex at T=4.2K

Current-voltage characteristics of annular junction

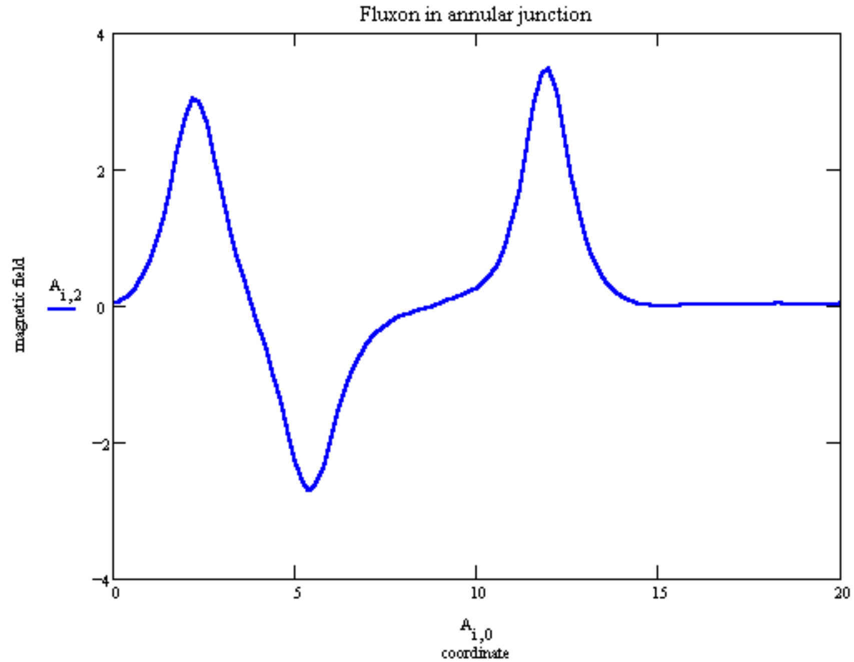


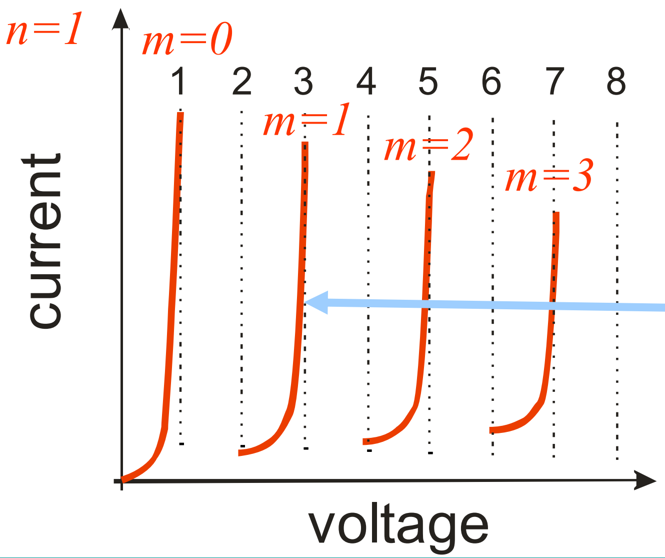
Multi-soliton dynamics in annular junction

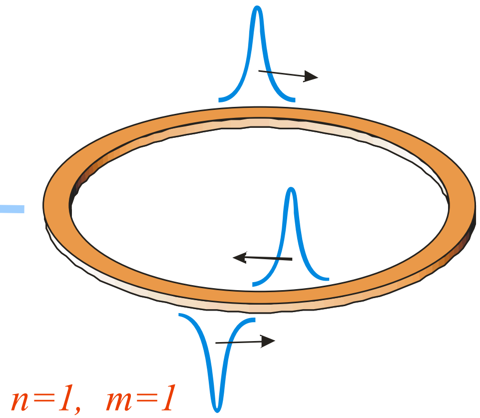
Example:

1 trapped vortex
(n=1)

+ 1 vortex-antivortex
pair *(m=1)*

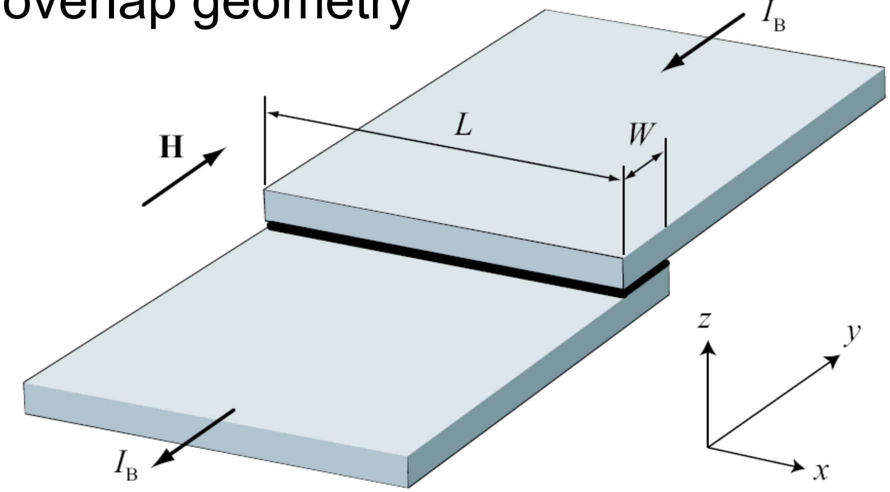




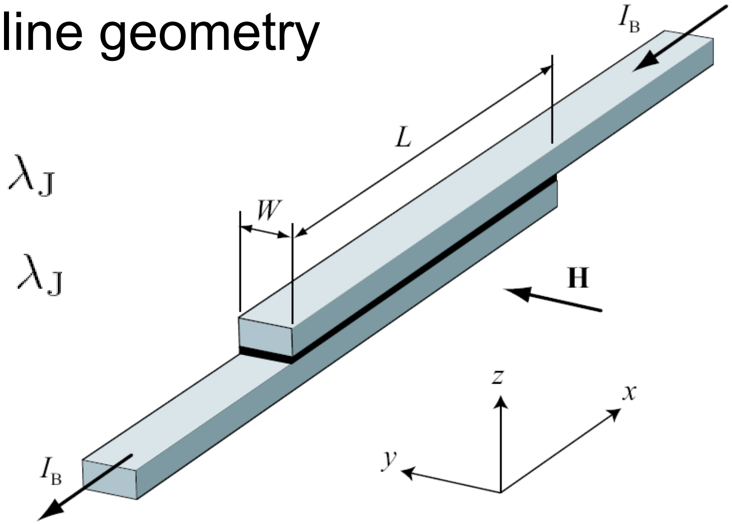


Long junction geometries

overlap geometry



inline geometry



$$L \gg \lambda_J$$

$$W \ll \lambda_J$$

$$\left.(\varphi_x + \beta\varphi_{xt})\right|_{x=0} = \eta\,,$$

$$\left.(\varphi_x + \beta\varphi_{xt})\right|_{x=\ell} = \eta\,,$$

where

$$\eta = \frac{2\pi\Lambda\lambda_J\mu_0}{\Phi_0}H\,.$$

$$\left.(\varphi_x + \beta\varphi_{xt})\right|_{x=0} = \eta + \kappa$$

$$\left.(\varphi_x + \beta\varphi_{xt})\right|_{x=\ell} = \eta - \kappa$$

where

$$\kappa = \frac{I_B}{2\lambda_J j_c W}\,.$$

SG Hamiltonian without perturbations

Equation $\varphi_{xx} - \varphi_{tt} = \sin\varphi$ can be attributed to a Hamiltonian system for $(\varphi,\ \varphi_t)$

$$\mathcal{H} \equiv \int_{-\infty}^{\infty} \left(\frac{1}{2}\varphi_x^2 + \frac{1}{2}\varphi_t^2 + 1 - \cos\varphi \right) \mathrm{d}x \ .$$

As $\varphi_x \to 0$ for $|x| \to \infty,$ we get:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} (\frac{1}{2}\varphi_x^2 + \frac{1}{2}\varphi_t^2 + 1 - \cos\varphi)\,\mathrm{d}x = \int_{-\infty}^{\infty} (\varphi_x\varphi_{xt} + \varphi_t\varphi_{tt} + \varphi_t \sin\varphi)\,\mathrm{d}x$$

$$= \varphi_x\varphi_t \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (-\varphi_t\varphi_{xx} + \varphi_t\varphi_{tt} + \varphi_t \sin\varphi)\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \varphi_t(-\varphi_{xx} + \varphi_{tt} + \sin\varphi)\,\mathrm{d}x = 0 \ .$$

For the unperturbed sine-Gordon system, the energy is conserved.

SG equation with perturbations

Perturbed sine-Gordon equation $\varphi_{xx} - \varphi_{tt} = \sin\varphi + \underbrace{\alpha\varphi_t - \beta\varphi_{xxt} - \gamma}_{\text{perturbations}} .$

From $\int\limits_{-\infty}^{\infty} \varphi_t(-\varphi_{xx} + \varphi_{tt} + \sin\varphi)\,\mathrm{d}x = 0$ we get:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int\limits_{-\infty}^{\infty} (-\alpha\varphi_t^2 + \beta\varphi_{xxt}\varphi_t + \gamma\varphi_t)\,\mathrm{d}x$$

$$= \beta\varphi_{xt}\varphi_t\Big|_{-\infty}^{\infty} + \int\limits_{-\infty}^{\infty} (-\alpha\varphi_t^2 - \beta\varphi_{xt}\varphi_{xt} + \gamma\varphi_t)\,\mathrm{d}x$$

$$= \int\limits_{-\infty}^{\infty} (\underbrace{-\alpha\varphi_t^2 - \beta\varphi_{xt}^2}_{\text{energy dissipation}} + \underbrace{\gamma\varphi_t}_{\text{energy input}})\,\mathrm{d}x .$$

Energy balance of a single soliton

$$\varphi_{\mathrm{F}} = 4\arctan\left[\exp\frac{x - vt - x_0}{\sqrt{1-v^2}}\right],$$ by inserting this into Hamiltonian we get

the soliton energy $\mathcal{H}(\varphi_{\mathrm{F}}) \equiv \mathcal{E}_{\mathrm{F}} = \dfrac{8}{\sqrt{1-v^2}}$ as that of relativistic particle with

rest mass $m_{\mathrm{F}} = 8.$ Since α, β and γ are assumed to be small, we take as an approximate solution of the perturbed equation the soliton-type solution given above with a *time-dependent velocity* v.

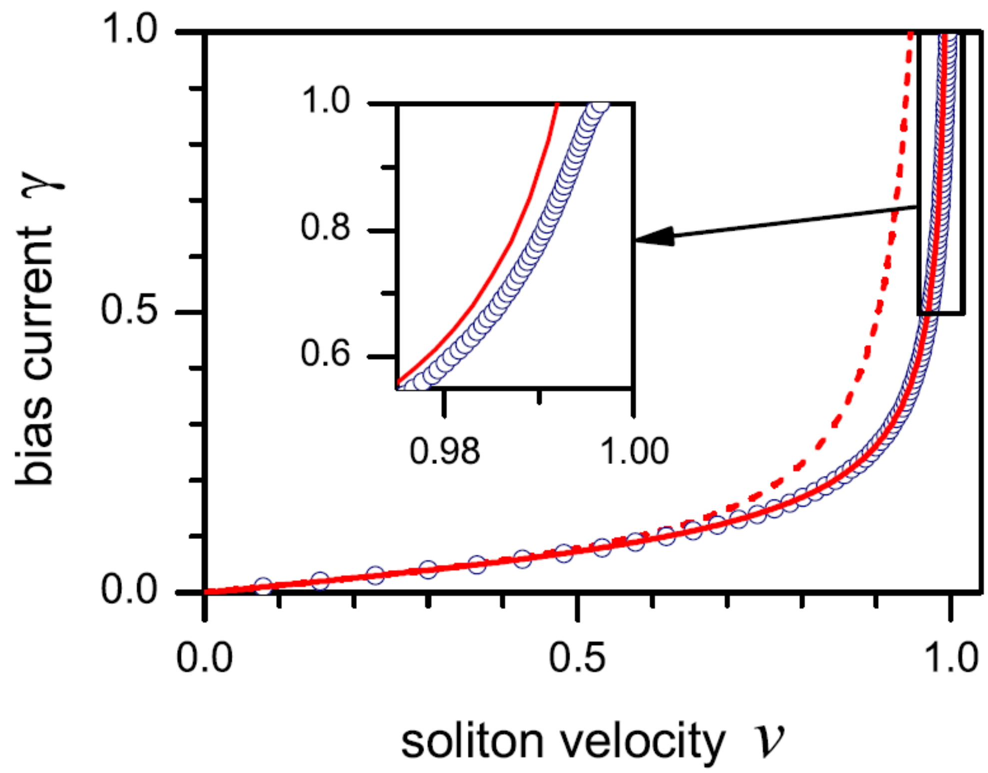
$$\left.\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}\right|_{\varphi=\varphi_{\mathrm{F}}} = -8\alpha\frac{v^2}{(1-v^2)^{1/2}} - \frac{8}{3}\beta\frac{v^2}{(1-v^2)^{3/2}} - 2\pi\gamma\,.$$

Using the last formula of the previous slide we get:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\alpha v(1-v^2) - \frac{1}{3}\beta v - \frac{1}{4}\pi\gamma(1-v^2)^{3/2}\,.$$

The *power balance velocity*: ⇨ $$\gamma = \frac{4}{\pi}\frac{|v_\infty|}{\sqrt{1-v_\infty^2}}\left[\alpha + \frac{\beta}{3(1-v_\infty^2)}\right]$$

Comparison with numerical simulations



Dependence of the power balance soliton velocity on the bias current with α = *0.08* and β = *0.06* (dashed line), and with α = 0.1 and β = 0 (solid).
Open dots show the result of full numerical simulations with α = 0.1 and β = 0.