Exercise 3 of the Lecture "Full waveform inversion"

FWI WITH STEEPEST DESCENT METHOD

January 10, 2022

Goal

In this exercise, we perform our first FWI synthetic test by using the steepest descent method.

Steepest descent method

FWI tries to find an optimal model v^{opt} which minimize the misfit function:

$$J(v) = \frac{1}{2} \sum_{s_r, x_r} \int_0^T \|\hat{p}^{syn}(t, x_r) - p^{obs}(t, x_r)\|^2 dt. \tag{1}$$

where $\hat{p}$ is the solution of the wave equation by using a model v. In the last exercise we arrived that the gradient of $J(v)$ with respect to v can be calculated by using the adjoint state method that

$$\frac{\partial J(v)}{\partial v} = \sum_{s_r} \int_0^T \hat{p}^{\dagger}(x, y, t' = T - t) \frac{\partial^2 \hat{p}(x, y, t)}{\partial t^2} dt. \tag{2}$$

where $\hat{p}^{\dagger}$ is the adjoint wavefield. The optimal model can be found by iteratively

$$v_{k+1} = v_k + \alpha \delta v_k \tag{3}$$

where α is the step length and δv_k is the model update direction at the kth iteration. In the case of steepest descent method, we directly choose the negative gradient direction as the model update direction that

$$\delta v_k = -\frac{\partial J(v)}{\partial v_k} \tag{4}$$

Due to the strong gradient located around the source points, a circular taper with a radius of 100 m is applied to the source position.

Step length estimation

The aim is to find an appropriate step length α to update the model.

Option 1: Constant step length

We can use a constant step length α that is proportional to the maximum value of the model to update the model iteratively.

$$v_{k+1} = v_k + \alpha \delta v_k \tag{5}$$

where

$$\alpha = p \times \frac{v^{max}}{max(|\delta v_k|)} \tag{6}$$

v^{max} is usually a predefined maximum value of model v and p is a scaling factor. In this exercise, we choose $v^{max} = max(v_k)$ and $p = 0.01$. We update the model iteratively regardless of whether the misfit value is increased or reduced until the maximum number of iteration (30 in this exercise) is reached.

Option 2: step length with line search

To solve this problem a variable step length is introduced. For two trial step lengths α_1 and α_2 and their corresponding test models $v_{k+1}^{(1)}$ and $v_{k+1}^{(2)}$ are calculated

$$v_{k+1}^{(1)} = v_k + \alpha_1 \delta v_k \tag{7}$$

$$v_{k+1}^{(2)} = v_k + \alpha_2 \delta v_k \tag{8}$$

where α_1 and α_2 are calculated by

$$\alpha_1 = p_1 \times \frac{max(v_k)}{max(|\delta v_k|)} \tag{9}$$

$$\alpha_2 = p_2 \times \frac{max(v_k)}{max(|\delta v_k|)} \tag{10}$$

Here we choose $p_1 = 0.01$ and $p_2 = 0.03$ as a simple example.
Then, we calculate a third trial model $v_{k+1}^{(3)} = v_k + \alpha_3 \delta v_k$ and its corresponding misfit value $J(v_{k+1}^{(3)})$ depending on the values of $J(v_{k+1}^{(1)})$ and $J(v_{k+1}^{(2)})$:
Case 1: if $J(v_k) < J(v_{k+1}^{(1)}) < J(v_{k+1}^{(2)})$, we calculate a third trial step length $\alpha_3 = 0.5 * \alpha_1$;
Case 2: if $J(v_k) > J(v_{k+1}^{(1)}) > J(v_{k+1}^{(2)})$, we calculate a third trial step length $\alpha_3 = 2 * \alpha_2$;
Case 3: if $J(v_k) > J(v_{k+1}^{(1)})$and $J(v_{k+1}^{(1)}) < J(v_{k+1}^{(2)})$, we calculate the third test step length α_3 via parabolic fitting that

$$\alpha_3 = 0.5 * \frac{J(v_k) * (\alpha_1^2 - \alpha_2^2) + J(v_{k+1}^{(2)}) * (-\alpha_1^2)}{J(v_k) * (\alpha_1 - \alpha_2) + J(v_{k+1}^{(1)}) * (\alpha_2) + J(v_{k+1}^{(2)}) * (-\alpha_1)}; \tag{11}$$

Therefore, we have three trial step lengths α_1, α_2 and α_3, their corresponding models $v_{k+1}^{(1)}$, $v_{k+1}^{(2)}$ and $v_{k+1}^{(3)}$, as well as their misfit values $J(v_{k+1}^{(1)})$, $J(v_{k+1}^{(2)})$ and $J(v_{k+1}^{(3)})$. We choose the best step length α_i ($i = 1$, 2 or 3) that provides the lowest misfit value $J(v_{k+1}^{(i)})$.

Numerical algorithm

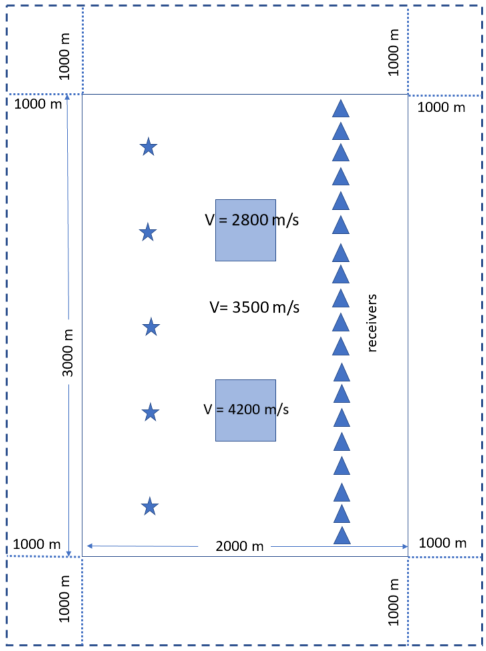
1. Loop over iterations k = 1, 2, ..., 30.
2. Calculate the gradient of current model δv_k.
3. Calculate the step length α for model updating.
4. Update the model via $v_{k+1} = v_k + \alpha \delta v_k$.

Exercise

1. Prepare a script "fwisd.m" to perform FWI with the steepest descent method. You may choose either a constant (option 1) or dynamic (option 2) step length for model update. (10 points)
2. Plot out the models after 5, 10, 20 and 30 iterations by using the function *imagesc*. (5 points)
3. Plot the evolution of data misfit (i.e., $J(v_k)$) by using the function *plot*. (3 points)
4. Plot the evolution of model misfit (i.e., $\|v_k - v_{true}\|^2$ where v_{true} represent the true model) by using the function *plot*. (2 points)

Model 1 (Initial homogeneous model)
Size of the model : 5000 (z) * 4000 (x) m
Source locations: (1500,1500), (2000,1500), (2500,1500), (3000,1500), (3500,1500) m
Receivers' location: (1200, 2500), (1250, 2500), (1300, 2500), ..., (3800, 2500) m
Velocity of the model: 3500 m/s
Density of the model: 2500 kg/m^3
Source time function: 10 Hz Ricker wavelet (with 0.3 s delay)
Total recording time: 1.200 s
Grid size: $dx = dz = 25$ m, $dt = 0.004$ s

Model 2 (True crosshole model)
Every parameter is the same as model 1, then adding two rectangular inclusions:
Inclusion 1: v= 2800 m/s (1800 m <= z <= 2200 m, 1800 m <= x <= 2200 m),
Inclusion 2: v= 4200 m/s (2800 m <= z <= 3200 m, 1800 m <= x <= 2200 m),



Reports and scripts

Name your report (in pdf format) and matlab scripts to thomas.bohlen@kit.edu before the end of **31. Jan. 2022**.