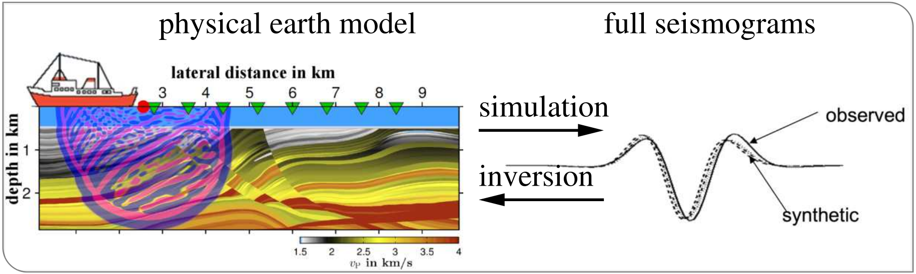
Elastic Full Waveform Inversion

Lecture 6 in WS 2020/21

Thomas Bohlen & Yudi Pan



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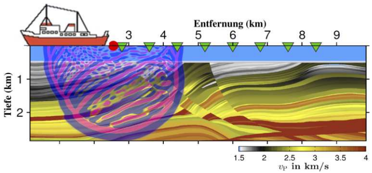
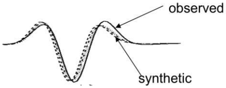
Goals of elastic FWI





P–waves
S–waves
Surface waves

Benefits

1. **Improved resolution:** $\approx \frac{\lambda}{2}$ ☺
2. **Multi-parameter reconstruction:**
   - a **P-wave velocity**
   - b **S-wave velocity**
   - c (Attenuation passive)
   - d (Anisotropy passive)
   - e **Density**
3. **Improved petrophysical characterization**

Elastic FWI

- Elastic forward and adjoint modeling
- required if S-waves or surfaces should be inverted

Advantages

1. Improved characterization by V_p and V_s
2. Improved resolution of S-waves

Challenges

1. Higher computational cost
2. Higher non-linearity
3. Parameter cross-talk

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Gradients for elastic FWI (adjoint method)

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The derivation of gradients for elastic FWI follows the same procedure as for the acoustic case:

1. Take elastic wave equation
2. Perturbation approach for model parameters and wave field properties gives
   a. wave equation for background fields
   b. wave equation for perturbed (scattered) fields
3. Formulation of wave equations in terms of Greens functions
4. Apply Born approximation ($\equiv$ Linearization of forward operator)
5. Identify Frechét derivatives $\frac{\partial u_i}{\partial m_i}$
6. Insert Frechét derivatives into formula for the gradient $\frac{\partial E}{\partial m_i}$

Detailed derivations can be found in (Köhn 2011, Köhn et al. 2012, Mora 1987). In the following I copied parts of slides prepared by Daniel Köhn (Kiel University) Köhn (2018).

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1. Elastic wave equation

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Elastic equations of motion for anisotropic media

$$\rho\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\sigma_{ij} = f_i,$$

$$\sigma_{ij} - c_{ijkl}\epsilon_{kl} = T_{ij},$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

+ initial and boundary conditions,

where ρ denotes the density, u_i the displacement, σ_{ij} the stress tensor, ϵ_{ij} the strain tensor, c_{ijkl} the stiffness tensor, f_i, T_{ij} source terms for volume and surface forces, respectively.

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2. Perturbation approach

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First order perturbations

In the next step every parameter and variable in the elastic wave equation is perturbated by a first order perturbation:

$$u_i \rightarrow u_i + \delta u_i,$$

$$\sigma_{ij} \rightarrow \sigma_{ij} + \delta\sigma_{ij}$$

$$\epsilon_{ij} \rightarrow \epsilon_{ij} + \delta\epsilon_{ij}$$

$$\rho \rightarrow \rho + \delta\rho$$

$$c_{ijkl} \rightarrow c_{ijkl} + \delta c_{ijkl}$$

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2. Perturbation approach

Pertubated elastic equations of motion

$$\rho\frac{\partial^2 \delta u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\delta\sigma_{ij} = \Delta f_i$$

$$\delta\sigma_{ij} - c_{ijkl}\delta\epsilon_{kl} = \Delta T_{ij}$$

$$\delta\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right)$$

$+$ perturbated initial and boundary conditions

The new source terms are

$$\Delta f_i = -\delta\rho\frac{\partial^2 u_i}{\partial t^2}, \qquad \Delta T_{ij} = \delta c_{ijkl}\epsilon_{kl}.$$

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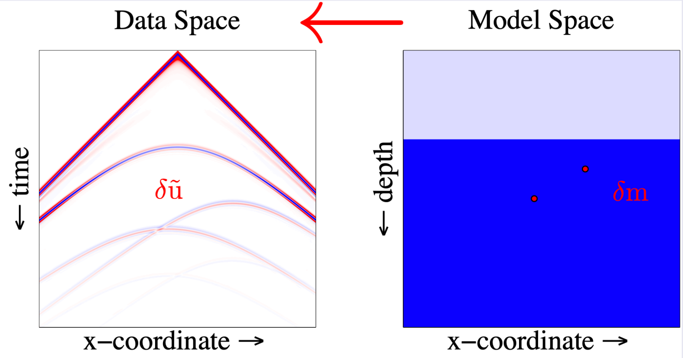
2. Perturbation approach

Pertubated elastic equations of motion



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3. Greens functions

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Definition: Green's function

If a unit impulse is applied as a source term at $\mathbf{x} = \mathbf{x}'$ at time $t = t'$ in the n-direction, then we denote the ith component of the displacement field at any point $(\mathbf{x}, t)$ as **Green's function** $G_{in}(\mathbf{x}, t; \mathbf{x}', t')$ ([Aki and Richards, 1980]).

$$\rho \frac{\partial^2 G_{in}}{\partial t^2} - \frac{\partial \sigma_{ij}}{\partial x_j} = \delta_{in}\delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}.$$

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3. Greens functions

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Solution of the perturbated wave equation in terms of Green's function

The solution of the perturbated elastic equations of motion in terms of the elastic Green's function $G_{ij}(\mathbf{x}, t; \mathbf{x}', t')$ can be written as:

$$\delta u_i(\mathbf{x}, t) = \int_V dV \int_0^T dt' G_{ij}(x, t; x', t') \Delta f_j(\mathbf{x}', t') - \int_V dV \int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(x, t; x', t') \Delta T_{jk}(\mathbf{x}', t').$$

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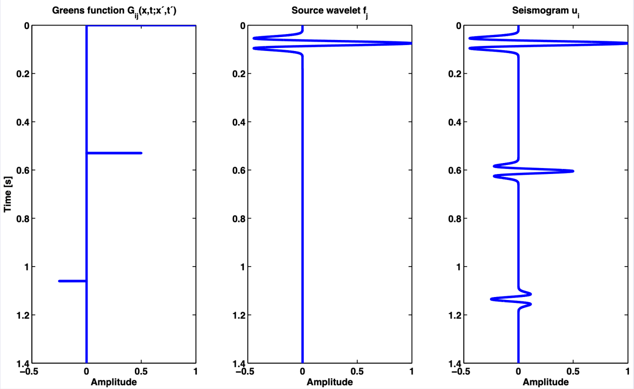
3. Greens functions

Simple example: Green's function



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3. Greens functions

Solution of the perturbated wave equation in terms of Green's function

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The new source terms are

$$\Delta f_j = -\delta\rho \frac{\partial^2 u_j}{\partial t^2}, \qquad \Delta T_{jk} = \delta c_{jklm} \epsilon_{lm}.$$

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3. Greens functions

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Subsitute source terms of the perturbated equations of motion

Substituting the force and traction source terms yields after some rearranging

$$\delta u_i(\mathbf{x},t) = -\int_V dV \int_0^T dt' G_{ij}(x,t;x',t') \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}',t')\delta\rho \\ -\int_V dV \int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(x,t;x',t')\epsilon_{lm}(\mathbf{x}',t')\delta c_{jklm}$$

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4. Born approximation

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Born approximation

Introducing isotropy via

$$\delta c_{jklm} = \delta_{jk}\delta_{lm}\delta\lambda + (\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})\delta\mu$$

leads to:

$$\begin{aligned}\delta u_i(\mathbf{x}, t) = & -\int_V dV \left[\int_0^T dt' G_{ij}(x, t; x', t') \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}', t')\right]\delta\rho \\ & -\int_V dV \left[\int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(x, t; x', t')\epsilon_{lm}(\mathbf{x}', t')\delta_{jk}\delta_{lm}\right]\delta\lambda \\ & -\int_V dV \left[\int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(x, t; x', t')\epsilon_{lm}(\mathbf{x}', t')(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})\right]\delta\mu.\end{aligned}$$

This equation has the same form as the desired expression for the forward problem:

$$\delta\mathbf{u} = \int_V dV \frac{\partial\mathbf{u}}{\partial\mathbf{m}}\delta\mathbf{m}.$$

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5. Frechét derivatives

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Identify Frechét kernels

Therefore the Frechét kernels $\frac{\partial u_i}{\partial \mathbf{m}(\mathbf{x})}$ for the individual material parameters can be identified as:

$$\frac{\partial u_i}{\partial \rho} = -\int_0^T dt' G_{ij}(x,t;x',t') \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}',t')$$

$$\frac{\partial u_i}{\partial \lambda} = -\int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(x,t;x',t') \epsilon_{lm}(\mathbf{x}',t') \delta_{jk}\delta_{lm}$$

$$\frac{\partial u_i}{\partial \mu} = -\int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(x,t;x',t') \epsilon_{lm}(\mathbf{x}',t') (\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})$$

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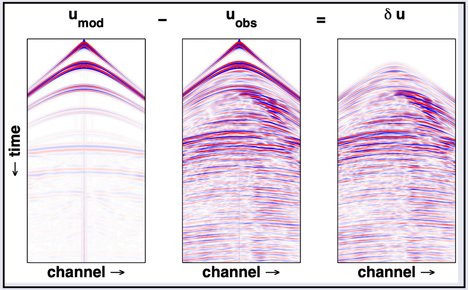
6. Gradient

Rewrite misfit function



$$E = \frac{1}{2}\delta\mathbf{u}^{\mathsf{T}}\delta\mathbf{u} = \frac{1}{2}\sum_{\text{sources}}\int dt \sum_{\text{receiver}} \delta\mathbf{u}^2(x_r, x_s, t)$$

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6. Gradient

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Gradient

To estimate the gradient direction $\partial \mathrm{E}/\partial \mathbf{m}$ the residual energy is rewritten as:

$$\mathrm{E} = \frac{1}{2}\delta \mathbf{u}^{\mathsf{T}}\delta \mathbf{u} = \frac{1}{2}\sum_{\text{sources}} \int \mathrm{dt} \sum_{\text{receiver}} \delta \mathbf{u}^2(\mathrm{x_r}, \mathrm{x_s}, \mathrm{t})$$

After derivation with respect to a model parameter **m** we get

$$\begin{aligned}\frac{\partial \mathrm{E}}{\partial \mathbf{m}} &= \sum_{\text{sources}} \int \mathrm{dt} \sum_{\text{receiver}} \frac{\partial \delta \mathbf{u}}{\partial \mathbf{m}} \delta \mathbf{u} \\ &= \sum_{\text{sources}} \int \mathrm{dt} \sum_{\text{receiver}} \frac{\partial (\mathbf{u}^{\mathbf{mod}}(\mathbf{m}) - \mathbf{u}^{\mathbf{obs}})}{\partial \mathbf{m}} \delta \mathbf{u} \\ &= \sum_{\text{sources}} \int \mathrm{dt} \sum_{\text{receiver}} \frac{\partial \mathbf{u}^{\mathbf{mod}}(\mathbf{m})}{\partial \mathbf{m}} \delta \mathbf{u}\end{aligned}$$

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6. Gradient

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$$\frac{\partial E}{\partial \rho} = -\sum_{\text{sources}} \int_0^T dt \sum_{\alpha=1}^{N_{rec}} \int_0^T dt' G_{ij}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}, t) \delta u_i'(x_\alpha, t')$$

$$\frac{\partial E}{\partial \lambda} = -\sum_{\text{sources}} \int_0^T dt \sum_{\alpha=1}^{N_{rec}} \int_0^T dt' \frac{\partial G_{ij}}{\partial x_k}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \epsilon_{lm}(\mathbf{x}, t) \delta_{jk} \delta_{lm} \delta u_i'(\mathbf{x}_\alpha, t')$$

$$\frac{\partial E}{\partial \mu} = -\sum_{\text{sources}} \int_0^T dt \sum_{\alpha=1}^{N_{rec}} \int_0^T dt' \frac{\partial G_{ij}}{\partial x_k}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \epsilon_{lm}(\mathbf{x}, t) (\delta_{jl} \delta_{km} + \delta_{jm} \delta_{kl}) \delta u_i'(\mathbf{x}_\alpha, t')$$

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6. Gradient

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Some rearrangements ...

The terms only depending on time t and the positions $\mathbf{x}$ can be moved infront of the sum over the receivers

$$\frac{\partial E}{\partial \rho} = -\sum_{\text{sources}} \int_0^T dt \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}, t) \sum_{\alpha=1}^{N_{rec}} \int_0^T dt' G_{ij}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \delta u_i'(x_\alpha, t'),$$

$$\frac{\partial E}{\partial \lambda} = -\sum_{\text{sources}} \int_0^T dt \epsilon_{lm}(\mathbf{x}, t) \delta_{jk} \delta_{lm} \sum_{\alpha=1}^{N_{rec}} \int_0^T dt' \frac{\partial G_{ij}}{\partial x_k}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \delta u_i'(\mathbf{x}_\alpha, t'),$$

$$\frac{\partial E}{\partial \mu} = -\sum_{\text{sources}} \int_0^T dt \epsilon_{lm}(\mathbf{x}, t) (\delta_{jl} \delta_{km} + \delta_{jm} \delta_{kl}) \sum_{\alpha=1}^{N_{rec}} \int_0^T dt' \frac{\partial G_{ij}}{\partial x_k}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \delta u_i'(\mathbf{x}_\alpha, t').$$

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6. Gradient

Introducing the wavefield Ψ_j

Defining the wavefield

$$\Psi_j(\mathbf{x},t) = \sum_{\alpha=1}^{N_{rec}} \int_0^T dt' G_{ij}(x_\alpha,t';\mathbf{x},t)\delta u_i'(x_\alpha,t'),$$

yields

$$\frac{\partial E}{\partial \rho} = -\sum_{sources} \int_0^T dt \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x},t)\Psi_j,$$

$$\frac{\partial E}{\partial \lambda} = -\sum_{sources} \int_0^T dt \epsilon_{lm}(\mathbf{x},t)\delta_{jk}\delta_{lm}\frac{\partial \Psi_j}{\partial x_k},$$

$$\frac{\partial E}{\partial \mu} = -\sum_{sources} \int_0^T dt \epsilon_{lm}(\mathbf{x},t)(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})\frac{\partial \Psi_j}{\partial x_k}.$$

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6. Gradient

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Calculate implicit sums ...

Writing out the implicit sums in the gradients of the Lamé parameters $\delta\lambda'$ and $\delta\mu'$

$$\frac{\partial E}{\partial \lambda} = -\sum_{\text{sources}} \int_0^T dt \sum_l \sum_k \sum_j \sum_m \epsilon_{lm}(\mathbf{x}, t)\delta_{jk}\delta_{lm}\frac{\partial \Psi_j}{\partial x_k},$$

$$\frac{\partial E}{\partial \mu} = -\sum_{\text{sources}} \int_0^T dt \sum_l \sum_k \sum_j \sum_m \epsilon_{lm}(\mathbf{x}, t)(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})\frac{\partial \Psi_j}{\partial x_k}.$$

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6. Gradient

... for the 2D PSV problem

Neglecting all wavefield components and derivatives in z-direction leads to

$$\frac{\partial E}{\partial \lambda} = -\sum_{\text{sources}} \int_0^T dt \left(\epsilon_{xx} + \epsilon_{yy}\right)\left(\frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right),$$

$$\frac{\partial E}{\partial \mu} = -\sum_{\text{sources}} \int_0^T dt \left[\left(\epsilon_{xy} + \epsilon_{yx}\right)\left(\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right)\right.$$
$$\left. + 2\left(\epsilon_{xx}\frac{\partial \Psi_x}{\partial x} + \epsilon_{yy}\frac{\partial \Psi_y}{\partial y}\right)\right].$$

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6. Gradient

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Introducing the strain tensor

Using the definition of the strain tensor ϵ_{ij} we get

$$\frac{\partial E}{\partial \lambda} = -\sum_{\text{sources}} \int_0^T dt \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\left(\frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right),$$

$$\frac{\partial E}{\partial \mu} = -\sum_{\text{sources}} \int_0^T dt \left[\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\left(\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right)\right]$$

$$+2\left(\frac{\partial u_x}{\partial x}\frac{\partial \Psi_x}{\partial x} + \frac{\partial u_y}{\partial y}\frac{\partial \Psi_y}{\partial y}\right).$$

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6. Gradient

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Final gradients

Finally the gradients for the Lamé parameters λ, μ and the density ρ can be written as

$$\frac{\partial E}{\partial \lambda} = -\sum_{\text{sources}} \int dt \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\left(\frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right)$$

$$\frac{\partial E}{\partial \mu} = -\sum_{\text{sources}} \int dt \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\left(\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right) + 2\left(\frac{\partial u_x}{\partial x}\frac{\partial \Psi_x}{\partial x} + \frac{\partial u_y}{\partial y}\frac{\partial \Psi_y}{\partial y}\right)$$

$$\frac{\partial E}{\partial \rho} = \sum_{\text{sources}} \int dt \left(\frac{\partial^2 u_x}{\partial t^2}\Psi_x + \frac{\partial^2 u_y}{\partial t^2}\Psi_y\right)$$

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FWI algorithm

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Full Waveform Tomography algorithm

1. For each shot solve the forward problem for the actual model m_n to generate a synthetic dataset u_{mod} and the wavefield $u_{mod}(\mathbf{x}, t)$.
2. Calculate the residual seismograms $\delta\mathbf{u} = \mathbf{u}^{\mathbf{mod}} - \mathbf{u}^{\mathbf{obs}}$.
3. Generate the wavefield $\boldsymbol{\Psi}(\mathbf{x}, t)$ by backpropagating the residuals from the receiver positions.
4. Calculate the gradients $\frac{\partial E}{\partial \mathbf{m}}$ for each material parameter.
5. Estimate the step length μ_n by a line search.
6. Update the material parameters using the gradient method $\mathbf{m}_{n+1} = \mathbf{m}_n - \mu_n \mathbf{P}_n \left(\frac{\partial E}{\partial \mathbf{m}}\right)_n$

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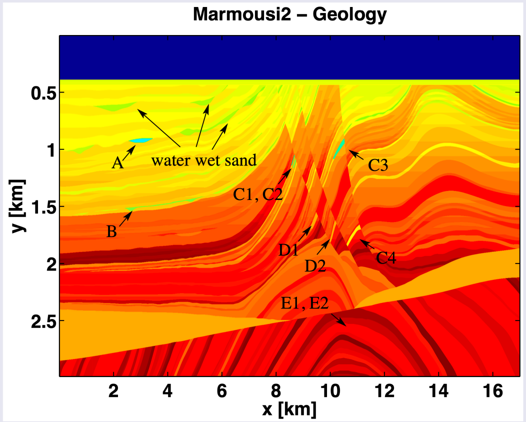
Marmousi model



Submarine thrust fault system with small hydrocarbon reservoirs (Martin et al. 2006)

- shallow gas (A)
- shallow oil (B)
- faulted trap gas sands (C1-C4)
- faulted trap oil sands (D1, D2)
- deep oil and gas (E1, E2)
- water wet sand

Acquisition geometry - acoustic sources and receivers

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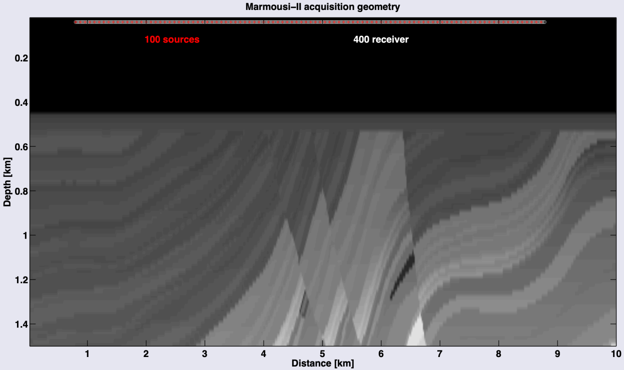
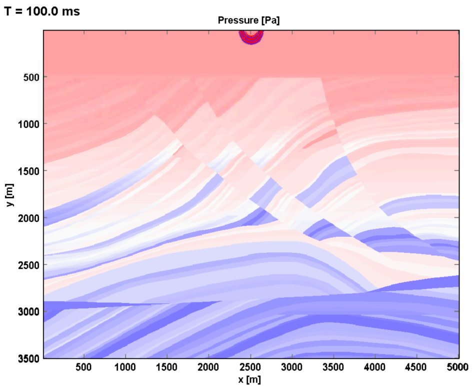
Marmousi-II acquisition geometry



Köhn (2011)

Wavefield

Propagation of the Pressure Wavefield



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Wavefield

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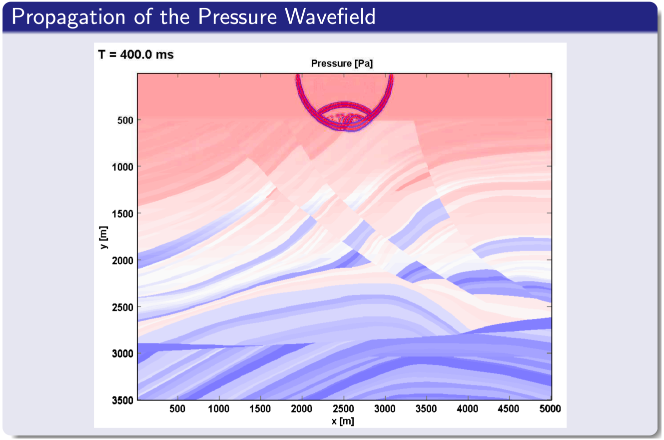

Köhn (2011)

Wavefield

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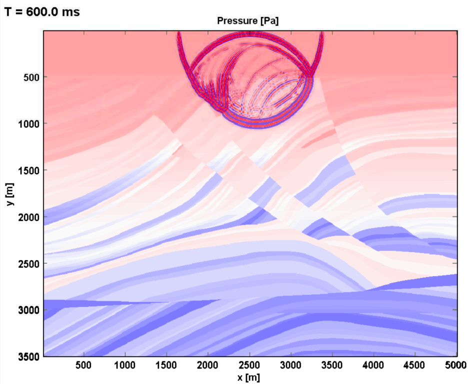
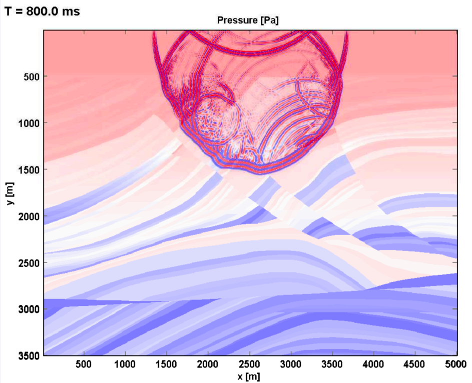


Köhn (2011)

Wavefield

Propagation of the Pressure Wavefield



Köhn (2011)

Wavefield

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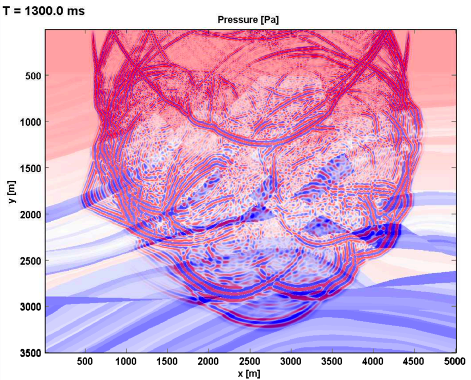
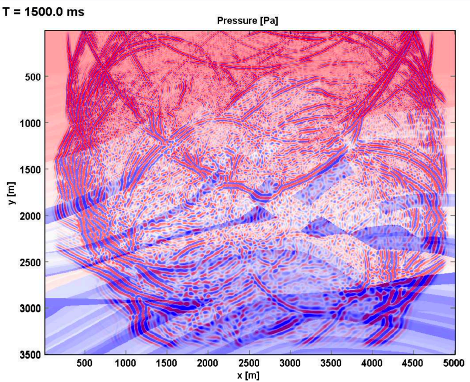


Köhn (2011)

Wavefield

Propagation of the Pressure Wavefield



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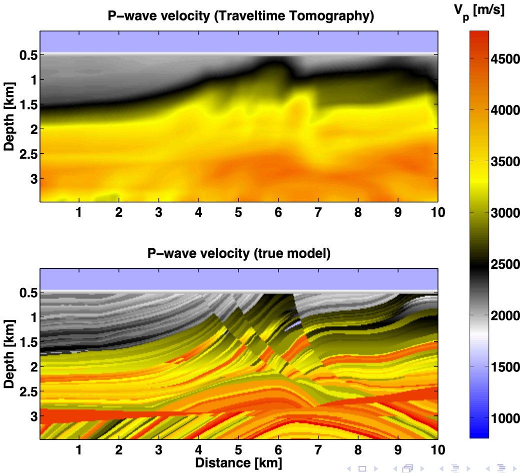
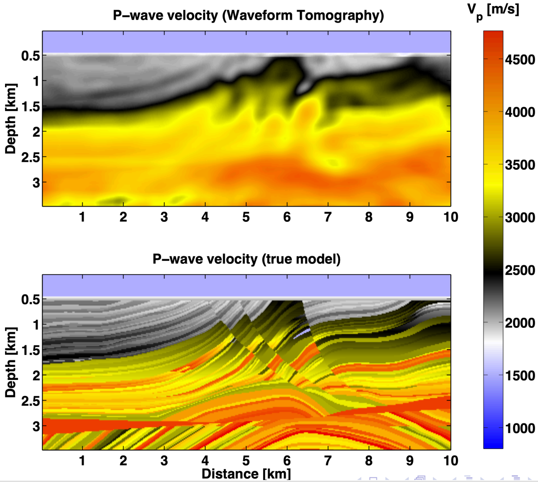
Initial and true model



Köhn (2011)

2Hz, 50 It.



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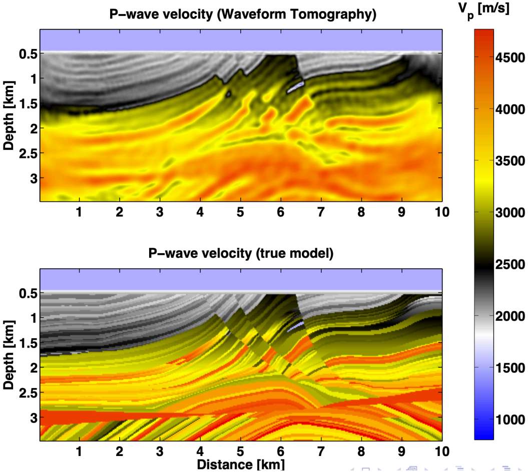
2-5 Hz, 75 It.



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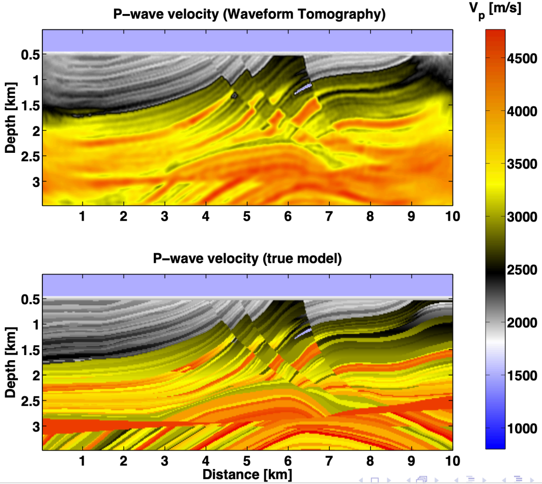
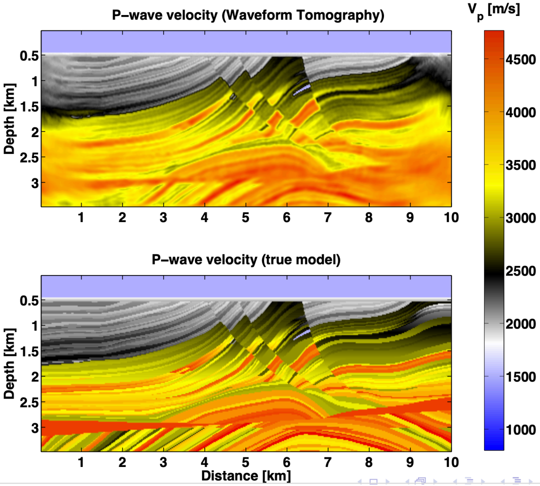
2-5-10 Hz, 90 It.



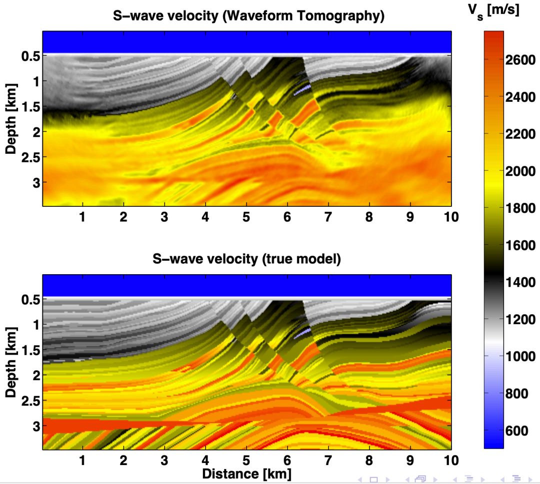
Köhn (2011)

2-5-10-20 Hz, 70 It.



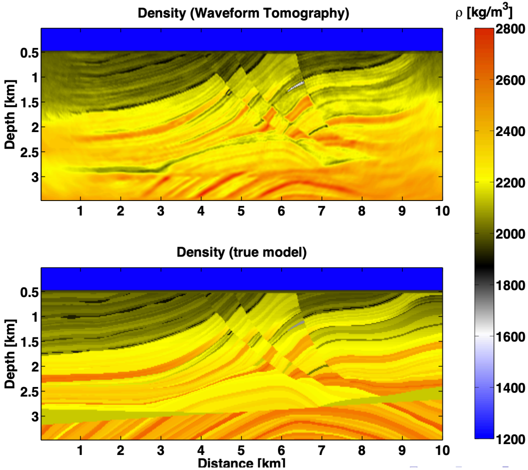
Köhn (2011)

2-5-10-20 Hz, 70 It.



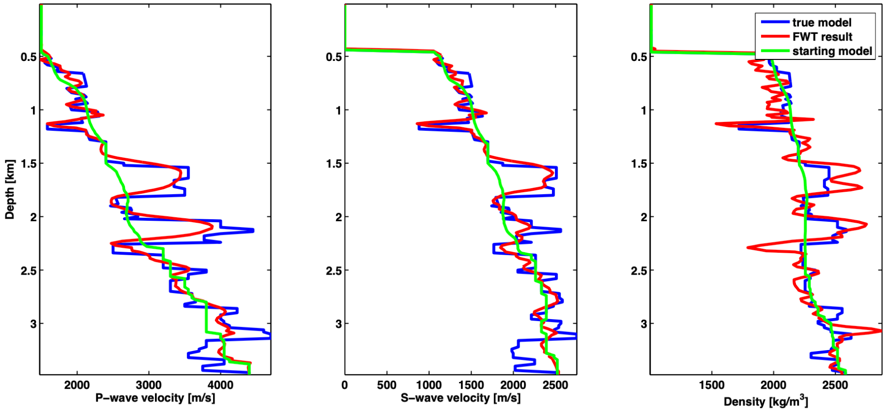
Köhn (2011)

2-5-10-20 Hz, 70 It.



Köhn (2011)

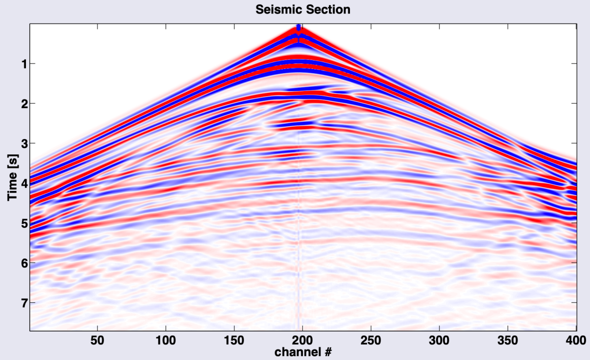
Vertical profile at x=6.4 km



Köhn (2011)

Shot gather

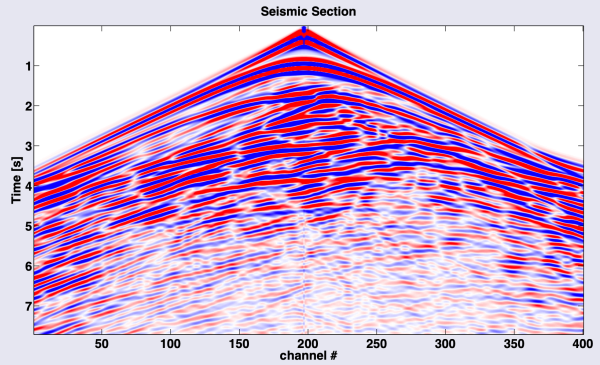
Seismic section for shot 50 (start model)



Köhn (2011)

Shot gather

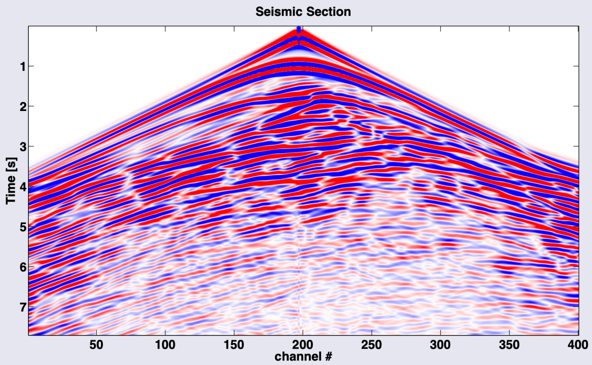
Seismic section for shot 50 (FWT result)



Köhn (2011)

Shot gather

Seismic section for shot 50 (true model)



Köhn (2011)

Agenda

Thank you for your attention

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