Full Waveform Inversion

Gradient calculation in FWI

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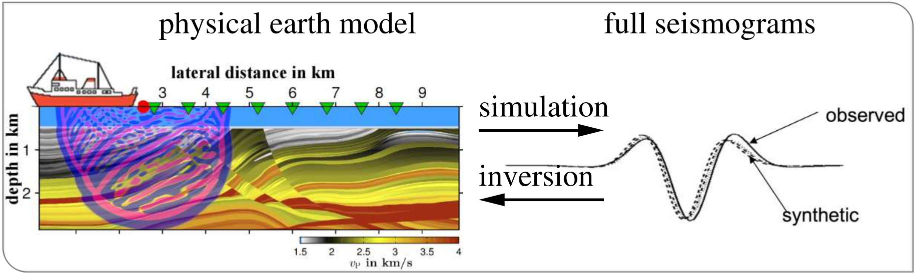

Agenda

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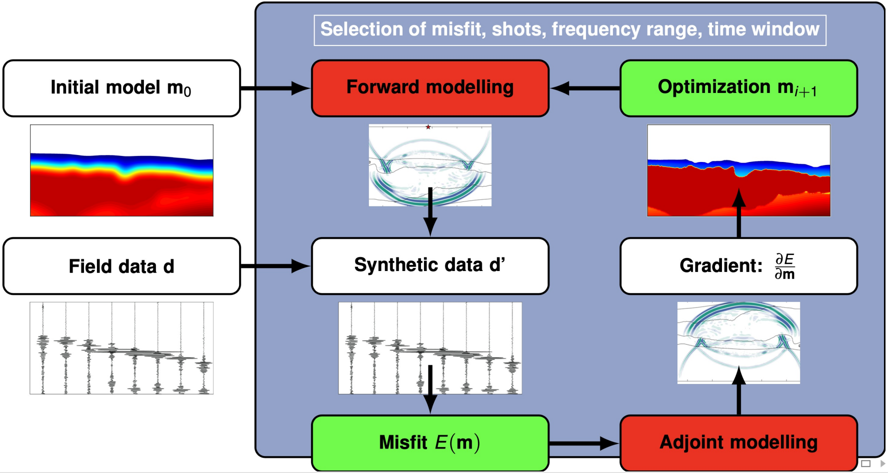
1. Summary of previous lecture

2. Calculation of Gradient
   2.1 Inversion of the single scattered wavefield
   2.2 Simple example

FWI Procedure



Summary of previous lecture

In the previous lecture we applied perturbation approach to model and data:

$$m = m_0 + \varepsilon m_1, \quad u(x) = u_0 + u_{sc} \tag{1}$$

We developed u_{sc} in a scattering series

$$u_{sc} = \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \tag{2}$$

We apply the *Born approximation* which assumes

1. $\epsilon \ll 1$: weak model perturbations m_1 and weak scattered field u_1
2. $u_{sc} = \varepsilon u_1$: single scattering only

Summary of previous lecture

Using the Born approximation we derived the following wave equations.

Background wavefield: $$m_0 \frac{\partial^2 u_0}{\partial t^2} - \Delta u_0 = f(x_s, t) \quad \text{or} \quad u_0(y) = G(y, x_s, t) f(x_s, t) \tag{3}$$

Scattered wavefield: $$m_0 \frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 = -m_1 \frac{\partial^2 u_0}{\partial t^2} \quad \text{or} \quad u_1(x_r) = -G(x_r, y, t) m_1 \frac{\partial^2 u_0}{\partial t^2} \tag{4}$$

Born approximation

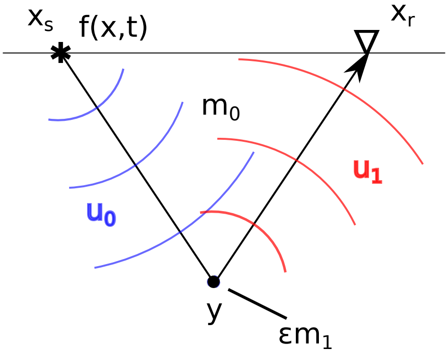


Figure: Born approximation: The arbitrary strong and complex background wavefield u_0 induced by source f is scattered only once at the weak model perturbations εm_1 in point y, producing the weak single scattered wavefield u_1.

Born approximation

Karlsruhe Institute of Technology

We insert 4 into 3

$$u_0(y,t) = G(y,x_s,t)f(x_s,t) \tag{5}$$

$$u_1(x_r,t) = -G(x_r,y,t)m_1\frac{\partial^2 u_0}{\partial t^2} = -m_1 G(x_r,y,t)\frac{\partial^2}{\partial t^2}(G(y,x_s,t)f(x_s,t)) \tag{6}$$

This is a *linear relation* between u_1 and m_1. The *Frechet-derivatives* (sensitivities):

$$\frac{\partial u_1(x_r,t)}{\partial m_1} = G(x_r,y,t)\frac{\partial^2}{\partial t^2}(G(y,x_s,t)f(x_s,t)) \tag{7}$$

They describe the change of scattered field $u_1(x_r,t)$ due to a model perturbation m_1 at y.
In the frequency domain we can write

$$\frac{\partial u_1(x_r,\omega)}{\partial m_1} = -\omega^2 G(x_r,y,\omega)G(y,x_s,\omega)f(x_s,\omega) \tag{8}$$

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1. Summary of previous lecture

2. Calculation of Gradient

2.1 Inversion of the single scattered wavefield

2.2 Simple example

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Outlook

1. In the following we define the misfit function $E(\omega, m)$ in the frequency domain.
2. We calculate the gradient $\frac{\partial E}{\partial m}$ by making use of the derived Frechet-derivatives, i.e. the Bornapproximation.
3. We will find an efficient procedure to calculate $\frac{\partial E}{\partial m}$.
4. We will transform the equation back into the time-domain.
5. We show one toy-example to illustrate the application.

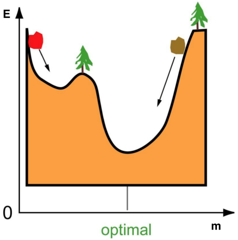
Inversion of the single scattered wavefield

We define the misfit function in the frequency domain as

$$E = \frac{1}{2}\int_{\omega} |u_1|^2 \, d\omega \tag{9}$$

Note that we used $u_1 \in \mathbb{C}$, i.e. the minimization of E aims to reduce the single scattered wavefield described by the Born approximation.
Our aim is to apply the steepest descent method:

$$m^{(n+1)} = m^{(n)} - \alpha \frac{\partial E}{\partial m}$$



Inversion of the single scattered wavefield

We calculate the gradient:

$$\frac{\partial E}{\partial m} = \frac{1}{2}\frac{\partial}{\partial m}\int_{\omega} \left| u_{1,R} + i u_{1,I} \right|^2 d\omega \tag{10}$$

The partial derivative of the complex function is calculated as follows:

$$\begin{aligned}
\frac{\partial}{\partial m}\left| u_{1,R} + i u_{1,I} \right|^2 = \frac{\partial}{\partial m}\left(u_{1,R}^2 + u_{1,I}^2\right) &= 2\left(u_{1,R}\frac{\partial u_{1,R}}{\partial m} + u_{1,I}\frac{\partial u_{1,I}}{\partial m}\right)\\
&= 2\Re\left[\left(\frac{\partial u_{1,R}}{\partial m} + i\frac{\partial u_{1,I}}{\partial m}\right)\left(u_{1,R} - i u_{1,I}\right)\right]\\
&= 2\Re\left[\frac{\partial u_1}{\partial m} u_1^*\right]
\end{aligned}$$

Gradient

Inserting this in equation (10) yields

$$\frac{\partial E}{\partial m} = \int_{\omega} \Re \left[\frac{\partial u_1}{\partial m} u_1^* \right] \, d\omega \tag{11}$$

We now use the Frechet-derivative

$$\frac{\partial u_1(x_r, \omega)}{\partial m_1} = -\omega^2 G(x_r, y, \omega) G(y, x_s, \omega) f(x_s, \omega) \tag{12}$$

leading to the misfit gradient

$$\frac{\partial E}{\partial m}(y) = \int_{\omega} \Re \left[-\omega^2 G(x_r, y, \omega) G(y, x_s, \omega) f(x_s, \omega) u_1^* \right] d\omega \tag{13}$$

Gradient

We re-arrange:

$$\boxed{\frac{\partial E}{\partial m}(y) = -\omega^2 \int_\omega \Re\left[\left(G(y, x_s, \omega) f(x_s, \omega)\right)\left(G(x_r, y, \omega) u_1^*\right)\right] d\omega} \tag{14}$$

Interpretation:

1. The first term $G(y, x_s, \omega) f(x_s, \omega)$ can be interpreted as *forward wavefield* from source point x_s to the image point y.
2. The second term $G(x_r, y, \omega) u_1^*$ can be interpreted as *backward residual wavefield* from the receiver point x_r to the image point y.
3. The integral describes a *zero-lag cross-correlation*.

1. Forward Wavefield

Remember that the term $G(y, x_s, \omega) f(x_s, \omega)$ actually describes solutions of the wave equation in the time-domain

$$m_0 \frac{\partial^2 u_f}{\partial t^2} - \Delta u_f = f(x_s, t) \tag{15}$$

The forward wavefield u_f is excited by $f(x_s, t)$ and propagates in the background model m_0.

2. Backward residual wavefield: $G(x_r, y, \omega) u_1^*$

1. The scattered field is defined as $u_1 = u - u_0$. u_1 can thus be interpreted as the residual field or missing wavefield not (yet) described by our background model.
2. The complex conjugation $u_1^* = u_{1,R} - iu_{1,I}$ implies a time reversal. Consider the Fourier-domain representation

$$u_1(\omega) = |u_1(\omega)| e^{i(\omega t + \varphi(\omega))}$$

   where $|u_1(\omega)|$ and $\varphi(\omega)$ denote the amplitude and phase spectrum, respectively. We see that

$$i \rightarrow -i \Leftrightarrow t \rightarrow -t$$

3. The general reciprocity relation of Greens-functions $G(x_r, y, \omega) = G(y, x_r, \omega)$. This implies that the residual wavefield can be excited at the receiver location x_r.

2. Backward residual wavefield: $G(x_r, y, \omega) u_1^*$

4 The term $G(y, x_r, \omega) u_1^*$ thus corresponds to the wave equation in the time-domain

$$m_0 \frac{\partial^2 u_b}{\partial t^2} - \Delta u_b = u_1(x_r, -t) \tag{16}$$

The backward wavefield u_b is excited by $u_1(x_r, -t)$.

5 The wave equation itself is insensitive to a time reversal $t \rightarrow -t$ as second order time derivatives $\frac{\partial}{\partial t^2}$ are applied, i.e. a time reversal will not change second order time derivatives. A time reversal of the source signal thus produces a wavefield that propagates backwards in time.

3. zero-lag cross-correlation

The frequency domain representation of the gradient

$$\frac{\partial E}{\partial m}(y) = -\omega^2 \int_\omega \Re\left[\left(G(y, x_s, \omega) f(x_s, \omega)\right)\left(G(x_r, y, \omega) u_1^*\right)\right] d\omega$$

describes a zero-lag crosscorrelation of the forward and backward wavefield.

Cross-correlation

In order to show the analogy we review the definition of cross-correlation between two real functions $x(t)$ and $y(t)$ in the time and frequency domain:

$$z(\tau) = (x(t) \times y(t))(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt$$

In the frequency domain a cross-correlation can be calculated via

$$Z(\omega) = X^*(\omega)Y(\omega)$$

where $Z(\omega)$, $X(\omega)$, $Y(\omega)$ are the Fourier transformations of $z(t)$, $x(t)$, $y(t)$, respectively. $X^*(\omega)$ denotes complex conjugation. The inverse Fourier transformation is

$$z(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)Y(\omega)e^{i\omega\tau}d\omega$$

Cross-correlation

The zero-lag cross-correlation for $\tau = 0$ can be written in the frequency domain as

$$z(\tau = 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)Y(\omega)e^{i\omega 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)Y(\omega)d\omega$$

If we compare this with our formula for the gradient

$$\frac{\partial E}{\partial m}(y) = -\omega^2 \int_\omega \Re\left[\left(G(y, x_s, \omega)f(x_s, \omega)\right)\left(G(x_r, y, \omega)u_1^*\right)\right]d\omega$$

we see that this indeed is similar to a zero-lag cross-correlation. We show later that in the time-domain the gradient is calculated via

$$\boxed{\frac{\partial E}{\partial m}(y) = \frac{\partial}{\partial t^2}\int_t \left(u_f(y, t)u_b(y, t)\right)dt} \tag{17}$$

Illustration of gradient calculation

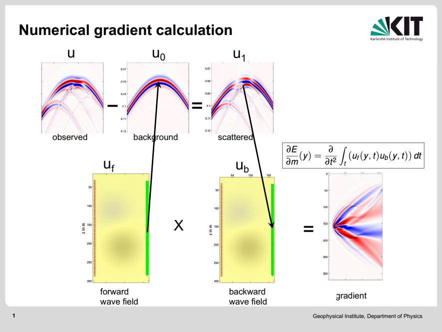
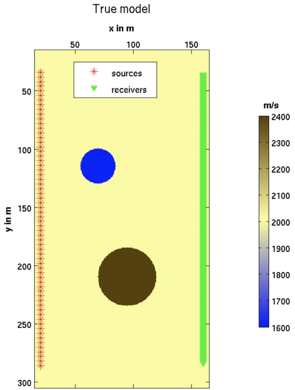


Figure: Gradient calculation: The gradient is obtained by zero-lag cross-correlation between the forward and backward wavefields. The forward field is excited by the source. The backward field is excited by the residuals injected at the receivers.

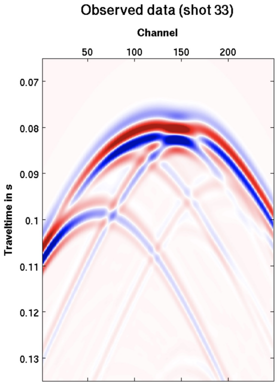
Agenda

Simple example



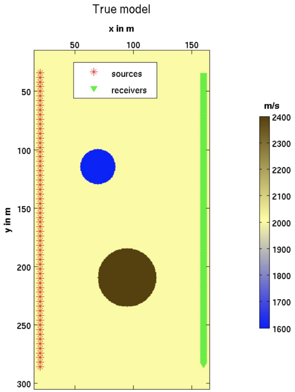
Forward

Problem



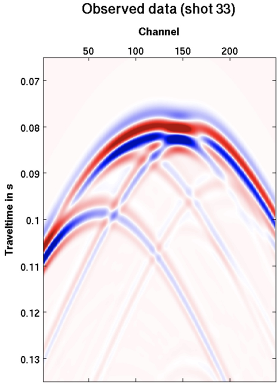
(Kurzmann 2012)

Simple example



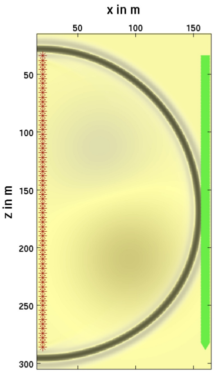
Inverse

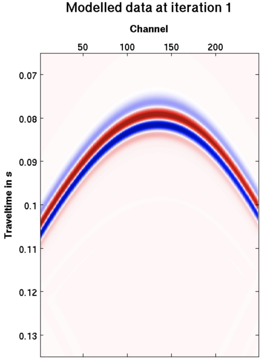
Problem



(Kurzmann 2012)

Acoustic forward modelling (movie)

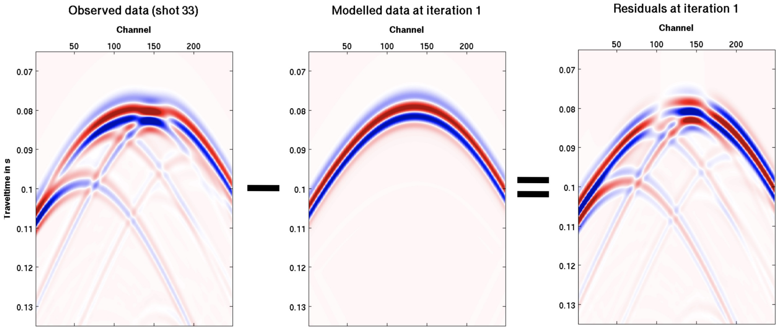




Starting model → **Synthetic forward data**

(Kurzmann 2012)

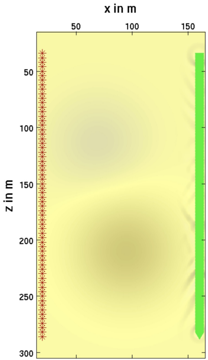
Calculation of residual seismograms and the misfit function



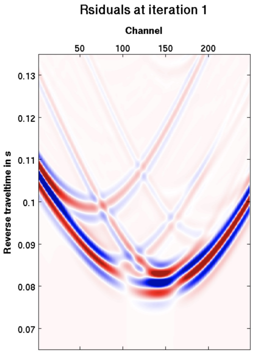
Observation **Synthetic data** **Data residuals**

(Kurzmann 2012)

Backpropagation (backprojection)(movie)



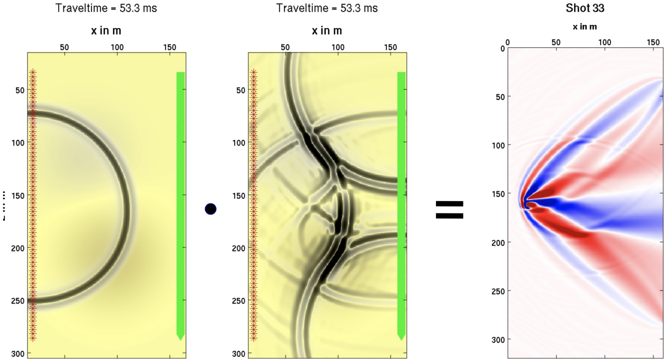
Starting model



- **Simultaneous input of data residuals at receiver positions in reverse time direction!**
- **Again: for every single shot!**

(Kurzmann 2012)

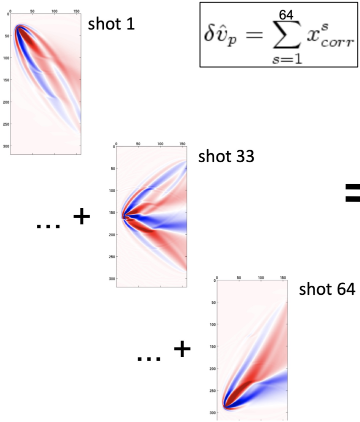
Gradient calculation = Imaging condition



u_j • ψ_j = $x^s_{corr} = \int \psi_j u_j dt$

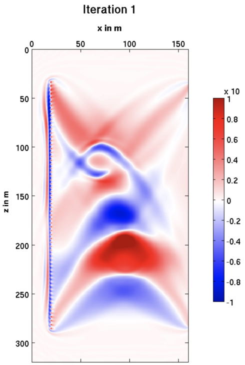
(Kurzmann 2012)

Gradient summation



$$\delta\hat{v}_p = \sum_{s=1}^{64} x_{corr}^{s}$$

=

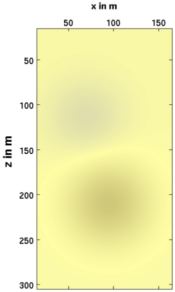


Gradient for all shots

(Kurzmann 2012)

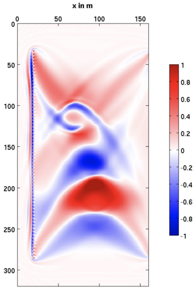
Gradient preconditioning and model update

Starting model



$+ \ \mu P \cdot$

Gradient



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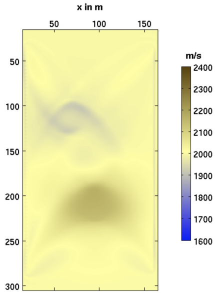
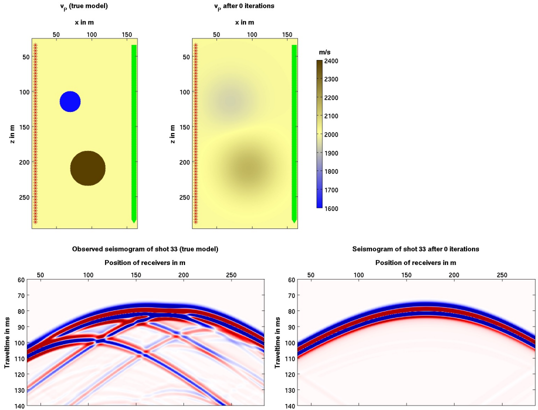
Updated model



(Kurzmann 2012)

Iterate until misfit has reached a minimum (movie)



(Kurzmann 2012)

Summary of lecture

- In the following we define the misfit function $E(\omega, m)$ in the frequency domain

$$E = \frac{1}{2}\int_{\omega} |u_1|^2 \, d\omega$$

- We calculated the gradient $\frac{\partial E}{\partial m}$

$$\frac{\partial E}{\partial m} = \int_{\omega} \Re\left[\frac{\partial u_1}{\partial m} u_1^*\right] \, d\omega$$

- We insert the Frechet-derivatives in the Bornapproximation and get

$$\boxed{\frac{\partial E}{\partial m}(y) = -\omega^2 \int_{\omega} \Re\left[\left(G(y, x_s, \omega) f(x_s, \omega)\right)\left(G(x_r, y, \omega) u_1^*\right)\right] d\omega}$$

  which describes a zero-lag cross-correlation between the forward and residual backward wavefield.
- We illustrated the steepest descent FWI in a simple acoustic crosswell experiment.

Thank you for your attention

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http://www.gpi.kit.edu/

References

Kurzmann, A. (2012), Applications of 2D and 3D full waveform tomography in acoustic and viscoacoustic complex media, PhD thesis, Karlsruhe Institute of Technology.
URL: *http://digbib.ubka.uni-karlsruhe.de/volltexte/1000034421*