Full Waveform Inversion

Adjoint method

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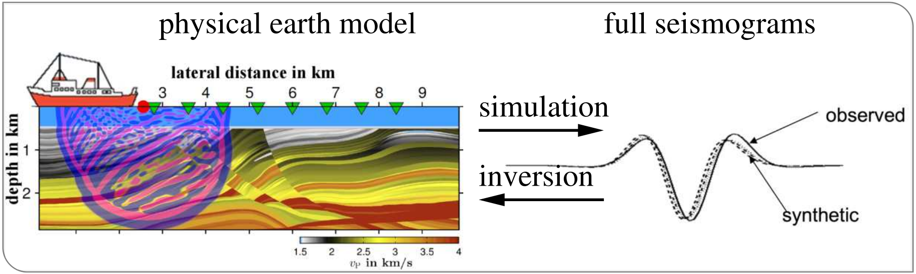

Agenda

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Agenda

1. Introduction
2. Linear forward operator for the wave equation
3. The adjoint of the linear forward operator
4. The adjoint method
5. The adjoint operator for the wave equation

Summary of lecture no. 2

- We defined the misfit function $E(\omega, m)$ in the frequency domain

$$E = \frac{1}{2} \int_\omega |u_1|^2 \, d\omega$$

- We calculated the gradient $\frac{\partial E}{\partial m}$

$$\frac{\partial E}{\partial m} = \int_\omega \Re \left[\frac{\partial u_1}{\partial m} u_1^* \right] \, d\omega$$

- We inserted the Frechet-derivatives in the Bornapproximation and obtained

$$\boxed{\frac{\partial E}{\partial m}(y) = -\omega^2 \int_\omega \Re \left[\left(G(y, x_s, \omega) f(x_s, \omega) \right) \left(G(x_r, y, \omega) u_1^* \right) \right] d\omega} \qquad (1)$$

  which describes a zero-lag cross-correlation between the forward and residual backward wavefield.
- We illustrated the steepest descent FWI in a simple acoustic crosswell experiment.

Outline

- Derive the formula 1 again in the time domain by using the adjoint state method
- The adjoint state method is a general formalism to derive gradients for linearized forward problems
- We first derive the linearized operator for the wave equation (again)
- We define the so-called adjoint operator and show how this relates to the gradient
- We derive the adjoint operator for the acoustic wave equation in the time domain
- We will see that the adjoint method provides a general mathematical framework for the gradient calculation.

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Linear forward operator

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We consider the acoustic wave equation

$$m\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t) \tag{2}$$

We define a corresponding forward operator that "solves" the wave equation

$$u = \tilde{F}[m] \tag{3}$$

We split our wavefield $u = u_0 + u_1$ up into background and scattered field. Our model parameters $m = m_0 + m_1$ into background model and perturbed model. The Taylor series of u around m_0 then reads

$$u = u_0 + \frac{\partial \tilde{F}}{\partial m}[m_0]m_1 + \frac{1}{2}\langle \frac{\partial^2 \tilde{F}}{\partial m^2}[m_0]m_1, m_2 \rangle \tag{4}$$

Linear forward operator

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We will consider only the first order term of this Taylor series. This corresponds to considering u_1 only, i.e single scattering (Born approximation). We define the linear operator

$$F := \frac{\partial \tilde{F}}{\partial m} = \frac{\partial u}{\partial m} \tag{5}$$

Compared with equation (4), this yields

$$u = u_0 + F m_1 \quad \Rightarrow \quad u_1 = F m_1 , \tag{6}$$

so that F describes a linear relation between u_1 and m_1, the Bornapproximation.

Linear forward operator

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To show the equivalence with the Born approximation, we differentiate the acoustic wave equation (28) with respect to m

$$\frac{\partial^2 u}{\partial t^2} + m \frac{\partial^2}{\partial t^2} \frac{\partial u}{\partial m} - \Delta \frac{\partial u}{\partial m} = 0 \tag{7}$$

Using (5) and (6), we replace $\frac{\partial u}{\partial m} = F = \frac{u_1}{m_1}$, resulting in

$$\frac{\partial^2 u}{\partial t^2} + (m_0 + m_1) \frac{\partial^2}{\partial t^2} \frac{u_1}{m_1} - \Delta \frac{u_1}{m_1} = 0$$

$$\Leftrightarrow \quad m_1 \frac{\partial^2}{\partial t^2} (u_0 + u_1) + (m_0 + m_1) \frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 = 0$$

Linear forward operator

In this equation, we can neglect the term $m_1 \frac{\partial^2 u_1}{\partial t^2}$, because those terms of higher order are describing multiple scattering:

$$m_0 \frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 = -m_1 \frac{\partial^2 u_0}{\partial t^2}, \tag{8}$$

which already resulted from the Born scattering theory. The solution for the scattered field u_1 can be expressed by the Green's function as

$$u_1 = -G_0 m_1 \frac{\partial^2 u_0}{\partial t^2} \tag{9}$$

This (the BA) is thus equivalent to the application of a linear forward operator

$$u_1 = F m_1 \tag{10}$$

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Adjoint operator

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If H_1 and H_2 are Hilbert spaces and the operator $T : \mathrm{H}_1 \rightarrow \mathrm{H}_2$ is linear, then the adjoint linear operator $T^* : \mathrm{H}_2 \rightarrow \mathrm{H}_1$ can be found with $x \in \mathrm{H}_1$ and $y \in \mathrm{H}_2$ via

$$\langle Tx, y\rangle_{H_2} = \langle x, T^* y\rangle_{H_1}, \tag{11}$$

where $\langle\rangle_{H_1}$ and $\langle\rangle_{H_2}$ denote scalar products in H_1 and H_2, respectively. In the integral notation of the scalar products, this means

$$\int_{H_2} Tx(r)\; y(r)\; dr_{H_2} = \int_{H_1} x(r)\; T^* y(r)\; dr_{H_1} \tag{12}$$

Adjoint operator

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We now apply this to our problem $u_1 = Fm_1$. The adjoint operator F^* thus fulfills the equation

$$\langle d_{obs}, \underbrace{Fm_1}_{u_1} \rangle_{\mathrm{D}} = \langle F^* d_{obs}, m_1 \rangle_{\mathrm{M}}, \tag{13}$$

The left side is a scalar product in data space D.
The right side one in model space M.

$F : \mathrm{M} \rightarrow \mathrm{D}$: forward operator
$F^* : \mathrm{D} \rightarrow \mathrm{M}$: *imaging* or *backward* operator

Adjoint operator

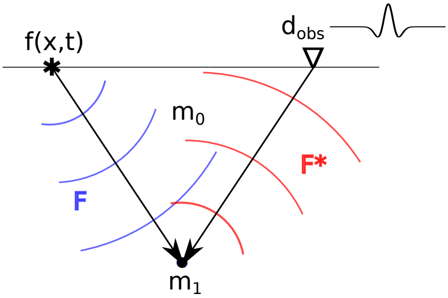

Figure: Interpretation of the forward operator F and its adjoint operator F^*

$$\langle d_{obs}, \underbrace{Fm_1}_{u_1} \rangle = \langle F^* d_{obs}, m_1 \rangle,$$

$F : \mathrm{M} \rightarrow \mathrm{D}$: forward operator

$F^* : \mathrm{D} \rightarrow \mathrm{M}$: backward operator

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Adjoint method

The Taylor expansion of the full operator $\tilde{F}$ gives

$$\tilde{F}[m+h] = \tilde{F}[m] + \frac{\partial \tilde{F}}{\partial m}[m]h + O(h^2) = \tilde{F}[m] + Fh + O(h^2) \tag{14}$$

We define the L2 misfit function

$$\begin{aligned} J(m) &= \frac{1}{2}||\tilde{F}[m] - d_{obs}||_2 \\ &= \frac{1}{2}\langle \tilde{F}[m] - d_{obs}, \tilde{F}[m] - d_{obs} \rangle \\ &= \frac{1}{2}\sum_{r,s}\int_0^T |\tilde{F}[m] - d_{obs}|^2 \, dt, \end{aligned} \tag{15}$$

$d_{obs} - \tilde{F}[m]$: data residual, $\sum_{r,s}$: summation over sources and receivers.

Adjoint method

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Inserting 14 into 15 yields

$$
\begin{aligned}
J(m+h) &= \frac{1}{2}\langle \tilde{F}[m+h] - d_{obs}, \tilde{F}[m+h] - d_{obs}\rangle \\
&= \frac{1}{2}\langle \tilde{F}[m] - d_{obs}, \tilde{F}[m] - d_{obs}\rangle + \langle Fh, \tilde{F}[m] - d_{obs}\rangle + O(h^2) \\
&= J(m) + \langle h, F^*(\tilde{F}[m] - d_{obs})\rangle + O(h^2) \qquad (16)
\end{aligned}
$$

We compare with the general Taylor expansion

$$
J(m+h) = J(m) + \langle h, \frac{\partial J}{\partial m}\rangle + O(h^2) \qquad (17)
$$

and find the **adjoint method**

$$
\boxed{\frac{\partial J}{\partial m} = F^*(\tilde{F}[m] - d_{obs})} \qquad (18)
$$

Adjoint method

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$$\boxed{\frac{\partial J}{\partial m} = F^*(\tilde{F}[m] - d_{obs})}$$

- The application of F^* corresponds to a back projection.
- The back projection of data residuals gives the gradient.
- The application of F^* is computationally very efficient. It avoids the explicit calculation of the Frechet derivatives $\frac{\partial u_1}{\partial m_1}$ which is much more expensive.
- The gradient calculation requires only to forward simulations:
  1. $u_{synth} = \tilde{F}[m]m$
  2. $\frac{\partial J}{\partial m} = F^*(u_{synth} - d_{obs})$
- Often $F = F^*$, i.e the same numerical solver can be used.

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F^* for the wave equation

We will now derive F^* for the acoustic wave equation. The starting point is

$$\underbrace{\langle d_{obs}, Fm_1 \rangle}_{\text{data space}} = \underbrace{\langle F^* d_{obs}, m_1 \rangle}_{\text{model space}} \tag{19}$$

We consider only one source but multiple receivers at x_r

$$d_{obs}(x,t) = \sum_r d_r(t)\delta(x - x_r) \tag{20}$$

With $Fm_1 = u_1$, equation (19) can be written as

$$\int_{\mathbb{R}^3} \int_0^T d_{obs}(x,t)\; u_1(x,t)\; dt\; dx = \int_{\mathbb{R}^3} F^* d_{obs}(x)\; m_1(x)\; dx \tag{21}$$

F^* for the wave equation

The relation $u_1 = Fm_1$ implies the two wave equations

$$\left(m_0 \frac{\partial^2}{\partial t^2} - \Delta\right) u_0 = f \tag{22}$$

$$\left(m_0 \frac{\partial^2}{\partial t^2} - \Delta\right) u_1 = -m_1 \frac{\partial^2 u_0}{\partial t^2}, \tag{23}$$

which are describing the generation and propagation of the background wavefield u_0 and the scattered wavefield u_1, respectively.

F^* for the wave equation

We now consider an auxiliary wavefield $q(x,t)$, the *adjoint wavefield*, that solves the wave equation with d_{obs} as right-hand side:

$$(m_0 \frac{\partial^2}{\partial t^2} - \Delta) q(x,t) = d_{obs}(x,t) \tag{24}$$

It can be interpreted as wavefield that is generated by the observed data as source. We substitute this into $\langle d_{obs}, Fm_1 \rangle$ and obtain

$$\begin{aligned} \langle d_{obs}, Fm_1 \rangle &= \int_{\mathbb{R}^3} \int_0^T (m_0 \frac{\partial^2}{\partial t^2} - \Delta) q(x,t) u_1(x,t) \; dt \; dx \\ &= \int_{\mathbb{R}^3} \int_0^T m_0 \frac{\partial^2 q}{\partial t^2} u_1 \; dt \; dx - \int_{\mathbb{R}^3} \int_0^T \Delta q \; u_1 \; dt \; dx \end{aligned} \tag{25}$$

F^* for the wave equation

We apply two times a partial integration in both space and time. The first integral is integrated in time, the second one in space. For the first one, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3}\int_0^T m_0 \frac{\partial^2 q}{\partial t^2} u_1 \; dt\, dx &= \int_{\mathbb{R}^3} m_0 \frac{\partial q}{\partial t} u_1 \Big|_0^T dx - \int_{\mathbb{R}^3}\int_0^T m_0 \frac{\partial q}{\partial t}\frac{\partial u_1}{\partial t} \; dt\, dx \\
&= \int_{\mathbb{R}^3} m_0 \frac{\partial q}{\partial t} u_1 \Big|_0^T dx - \int_{\mathbb{R}^3} m_0 q \frac{\partial u_1}{\partial t} \Big|_0^T dx + \int_{\mathbb{R}^3}\int_0^T m_0 q \frac{\partial^2 u_1}{\partial t^2} \; dt\, dx
\end{aligned}
\tag{26}
$$

F^* for the wave equation

For the second one, we use the theorem of Green, which transforms a volume integral into a surface integral via

$$\int_V (u_1 \Delta q - q \Delta u_1) \; dx = \int_{\partial V} \left(u_1 \frac{\partial q}{\partial n} - q \frac{\partial u_1}{\partial n} \right) \; dS_x, \tag{27}$$

where ∂V is the surface of the volume V, $\frac{\partial q}{\partial n}$ is the derivative in direction of the normal vector of the surface and dS_x means integration over the whole surface (Bronštejn 2015). With this theorem, the second integral in equation (25) can be written as

$$\int_V \int_0^T \Delta q \; u_1 \; dt \; dx = \int_V \int_0^T q \Delta u_1 \; dt \; dx + \int_{\partial V} \int_0^T u_1 \frac{\partial q}{\partial n} \; dS_x \; dt - \int_{\partial V} \int_0^T q \frac{\partial u_1}{\partial n} \; dS_x \; dt, \tag{28}$$

where we substituted the space integration over $\mathbb{R}^3$ with a finite volume V. V can be chosen arbitrary large.

F^* for the wave equation

Together with (26), the result for the scalar product is

$$\langle d_{obs}, Fm_1\rangle = \int_V \int_0^T q\left(m_0 \frac{\partial^2}{\partial t^2} - \Delta\right) u_1 \; dx \; dt + \int_V m_0 \frac{\partial q}{\partial t} u_1 \Big|_0^T dx - \int_V m_0 q \frac{\partial u_1}{\partial t}\Big|_0^T dx$$
$$- \int_{\partial V} \int_0^T u_1 \frac{\partial q}{\partial n} \, dS_x \; dt + \int_{\partial V} \int_0^T q \frac{\partial u_1}{\partial n} \, dS_x \; dt \qquad (29)$$

This can be simplified significantly by introducing boundary conditions. Because we chose the volume V so large that it covers nearly the space $\mathbb{R}^3$, it is impossible for the waves to reach the boundary of this volume in the considered travel time T. Thus, the integrations over the boundary of V can be neglected.

F^* for the wave equation

We use the starting conditions $u(t=0)=0$ and $\frac{\partial u}{\partial t}(t=0)=0$ and the final conditions $q(t=T)=0$ and $\frac{\partial q}{\partial t}(t=T)=0$. *The latter two conditions are fulfilled (and the integrals vanish) only when the $q(t)$ propagates backwards in time.* Then, equation (29) becomes

$$\begin{aligned}\langle d_{obs}, Fm_1\rangle &= \int_V \int_0^T q(x,t)\left(m_0 \frac{\partial^2}{\partial t^2} - \Delta\right) u_1(x,t)\, dx\, dt \\ &= -\int_V \int_0^T q(x,t) m_1 \frac{\partial^2 u_0}{\partial t^2}\, dx\, dt, \end{aligned} \tag{30}$$

where we used equation (23) for the last step. Because of $\langle d_{obs}, Fm_1\rangle = \langle F^* d_{obs}, m_1\rangle$, this gives us the **RTM imaging condition**

$$\boxed{F^* d_{obs} = -\int_0^T q(x,t) \frac{\partial^2 u_0}{\partial t^2}(x,t)\, dt} \tag{31}$$

F^* for the wave equation

With this we can calculate the gradient

$$\frac{\partial J}{\partial m}(m) = F^*(\tilde{F}[m] - d_{obs}) \tag{32}$$

We introduce the adjoint source f_{adj} in the adjoint equation (24)

$$(m_0 \frac{\partial^2}{\partial t^2} - \Delta)q(x,t) = f_{adj} \tag{33}$$

By comparison with 31 we see
$f_{adj} = d_{obs}$: RTM imaging condition, $f_{adj} = \tilde{F}[m] - d_{obs}$: gradient

$$\boxed{\frac{\partial J}{\partial m}(m) = F^*(\tilde{F}[m] - d_{obs}) = -\int_0^T q(x,t) \frac{\partial^2 u_0}{\partial t^2}(x,t)\, dt} \tag{34}$$

where $q(x,t)$ now corresponds to the backward propagated residual wave field.

Summary

- We introduced the adjoint method which is equivalent to the linear perturbation approach (BA)

$$\boxed{\frac{\partial J}{\partial m} = F^*(\tilde{F}[m] - d_{obs})}$$

- The adjoint operator is defined by

$$\underbrace{\langle d_{obs}, Fm_1 \rangle}_{\text{data space}} = \underbrace{\langle F^* d_{obs}, m_1 \rangle}_{\text{model space}}$$

- Explicit calculation for the wave equation

$$m\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t)$$

-

$$\boxed{\frac{\partial J}{\partial m}(m) = F^*(\tilde{F}[m] - d_{obs}) = -\int_0^T q(x,t)\frac{\partial^2 u_0}{\partial t^2}(x,t)\, dt}$$

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Thank you for your attention

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http://www.gpi.kit.edu/

References

Bronštejn, I. N. (2015), *Handbook of Mathematics*, 6th edn, Springer, Berlin, Heidelberg.