# Full waveform inversion of ground-penetrating radar data

## Lecture 9 of Full Waveform Inversion, WS 2021/22

Tan Qin and Thomas Bohlen

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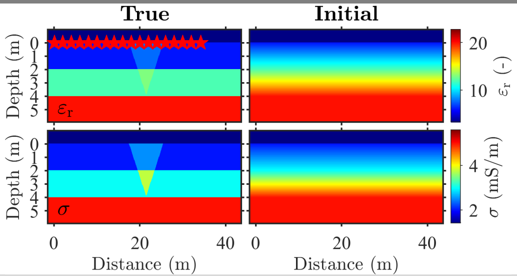
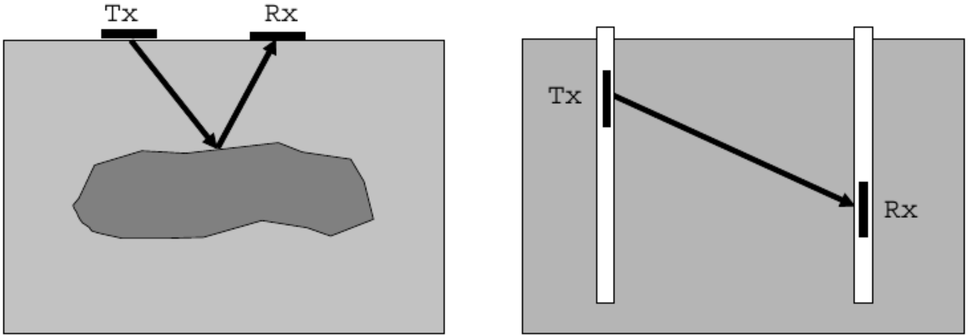


# Outline

# 1.1 Multi-offset GPR

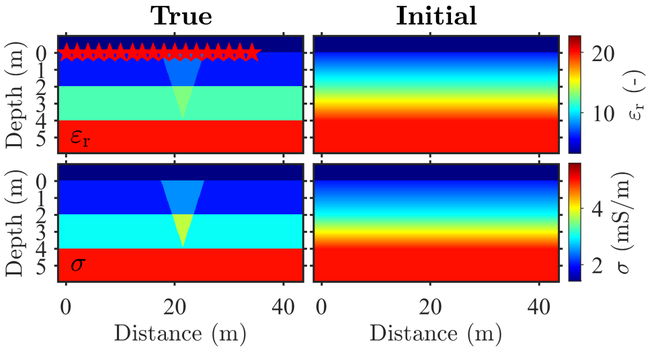
Multi-offset ground-penetrating radar (GPR) measurement on site.

## 1.1 Multi-offset GPR



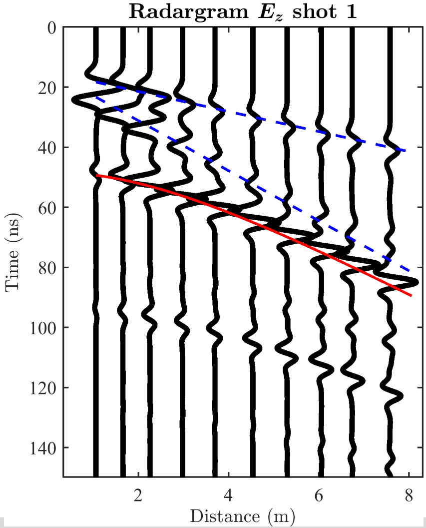

Two typical GPR data acquisitions (Annan, 2005). In this lecture, we will focus on surface-based GPR data. For crosshole GPR FWI, readers are recommended to Klotzsche et al. (2019) for details.

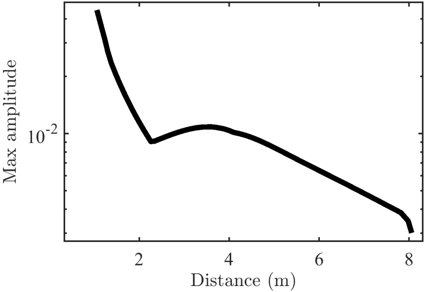
# 1.1 Multi-offset GPR




True model and initial model used in the synthetic inversion example of this lecture. $\varepsilon_r$ and $\sigma$ are the relative dielectric permittivity and electric conductivity, respectively. The red stars represent the transmitters (sources). The receivers are placed to the right of the transmitter with an offset of 1–8 m.

# 1.1 Multi-offset GPR

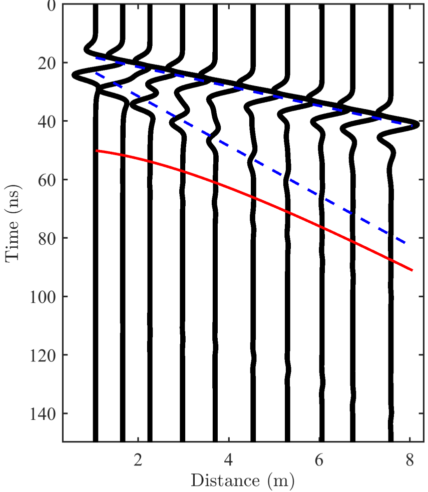


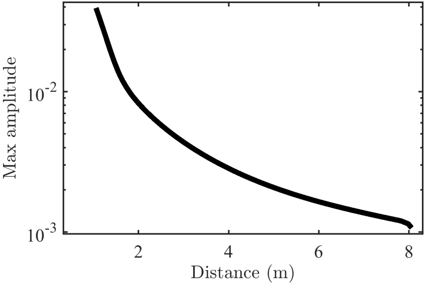




Radargram generated by a 100 MHz Ricker wavelet source on the surface of the true model and recorded by a receiver array on the surface. Data is normalized for better visualization.

# 1.1 Multi-offset GPR

**Radargram $E_z$ shot 1**




Radargram generated by a 100 MHz Ricker wavelet source on the surface of the initial model and recorded by a receiver array on the surface.


100 MHz Ricker wavelet

# Outline

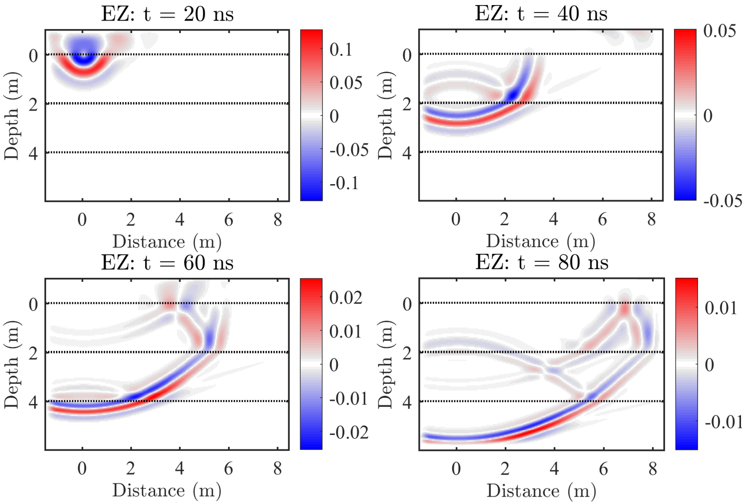
## 2.1 Forward problem

Maxwell's equations including constitutive relations are shown as two rotation equations:

$$\begin{cases} -\mu \dfrac{\partial \mathbf{H}}{\partial t} - \nabla \times \mathbf{E} = \mathbf{J_m} \\ \varepsilon \dfrac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} - \nabla \times \mathbf{H} = -\mathbf{J_e} \end{cases} \tag{2.1}$$

where $\sigma$, $\varepsilon$ and $\mu$ are the electric conductivity, dielectric permittivity and magnetic permeability, respectively. $\mathbf{J_e}$ and $\mathbf{J_m}$ are the electric and magnetic sources. $\mathbf{H}$ and $\mathbf{E}$ are the magnetic intensity and electric intensity in Maxwell's equations which explain the physical mechanism of electromagnetic (EM) wave propagation.
Similar to seismic waves, eq. 2.1 can be solved by the finite-difference time-domain (FDTD) method (Yee, 1966).

## 2.1 Forward problem




Snapshots of the electric wavefield propagating in the true model ($x$-$y$ plane where $x$ is the distance axis and $y$ is the depth axis).

## 2.1 Forward problem

3D EM wave equations in isotropic media:

$$\begin{cases} -\mu\frac{\partial H_x}{\partial t} - (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) = J_{mx} \\ -\mu\frac{\partial H_y}{\partial t} - (\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) = J_{my} \\ -\mu\frac{\partial H_z}{\partial t} - (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) = J_{mz} \\ \varepsilon\frac{\partial E_x}{\partial t} + \sigma E_x - (\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}) = -J_{ex} \\ \varepsilon\frac{\partial E_y}{\partial t} + \sigma E_y - (\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}) = -J_{ey} \\ \varepsilon\frac{\partial E_z}{\partial t} + \sigma E_z - (\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}) = -J_{ez} \end{cases} \tag{2.2}$$

For convenience, we rewrite eq.2.2 as:

$$\begin{cases} M_1\partial_t\mathbf{u} + M_2\mathbf{u} - Q\mathbf{u} = \mathbf{s} \\ \mathbf{u} = (\mathbf{H}, \mathbf{E})^T, \mathbf{s} = (\mathbf{J_m}, -\mathbf{J_e})^T \end{cases} \tag{2.3}$$

## 2.1 Forward problem

$$M_1 = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma \end{pmatrix}$$

## 2.1 Forward problem

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$$
\begin{cases}
Q = \begin{pmatrix} 0 & 0 & 0 & 0 & -\partial_z & \partial_y \\ 0 & 0 & 0 & \partial_z & 0 & -\partial_x \\ 0 & 0 & 0 & -\partial_y & \partial_x & 0 \\ 0 & -\partial_z & \partial_y & 0 & 0 & 0 \\ \partial_z & 0 & -\partial_x & 0 & 0 & 0 \\ -\partial_y & \partial_x & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \\
D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, D_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
D = D_1\partial_x + D_2\partial_y + D_3\partial_z, \quad D_i^* = D_i^T = -D_i
\end{cases}
\tag{2.4}
$$

* is the transpose conjugate operation.

## 2.2 Inverse problem

The objective function

$$\Phi_1(\mathbf{m}) = \frac{1}{2}||\mathbf{d}^{syn}(\mathbf{m}) - \mathbf{d}^{obs}||_2^2 = \frac{1}{2}||R\mathbf{u}(\mathbf{m}) - \mathbf{d}^{obs}||_2^2 \tag{2.5}$$

where the synthetic data $\mathbf{d}^{syn}$ is extracted by the restriction operator $R$ from the synthetic wavefield $\mathbf{u}$. The observed data $\mathbf{d}^{obs} = (\mathbf{H}^{obs}, \mathbf{E}^{obs})^T$.
Augmented functional used in the 1st order adjoint-state method (Plessix, 2006):

$$\begin{aligned} L_1(\mathbf{m}, \mathbf{u}, \mathbf{u}_1) &= \Phi_1(\mathbf{u}) + \langle \mathbf{u}_1, F(\mathbf{u}, \mathbf{m}) \rangle_W \\ F(\mathbf{u}, \mathbf{m}) &= M_1 \partial_t \mathbf{u} + M_2 \mathbf{u} - Q\mathbf{u} - \mathbf{s} = 0 \\ \mathbf{m} &= (\varepsilon, \sigma, \mu)^T, \mathbf{u}|_{t=0} = 0, \mathbf{u}|_{x \in \partial\Omega} = 0 \end{aligned} \tag{2.6}$$

where the inner product $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle_W$ in the domain $W = \Omega \times [0, T]$ is defined by ($\Omega$ is the spatial computation domain):

$$\langle \mathbf{h}_1, \mathbf{h}_2 \rangle_W = \int_0^T \int_\Omega \mathbf{h}_1^*(x, t) \mathbf{h}_2(x, t) dt dx \tag{2.7}$$

## 2.2 Inverse problem

For any $\mathbf{u}_1$:

$$L_1(\mathbf{m}, \mathbf{u}, \mathbf{u}_1) = \Phi_1(\mathbf{u}) \tag{2.8}$$

We write the augmented functional explicitly:

$$L_1(\mathbf{m}, \mathbf{u}, \mathbf{u}_1) = \frac{1}{2}\int_0^T \int_\Omega (R\mathbf{u} - \mathbf{d}^{obs})^2 dtdx + \int_0^T \int_\Omega \mathbf{u}_1^* (M_1 \partial_t \mathbf{u} + M_2 \mathbf{u} - Q\mathbf{u} - \mathbf{s}) dtdx \tag{2.9}$$

If the final condition and boundary condition of $\mathbf{u}_1$ are satisfied by:

$$\mathbf{u}_1|_{t=T} = 0, \mathbf{u}_1|_{x \in \partial\Omega} = 0 \tag{2.10}$$

Integration by parts and then we get:

$$\begin{aligned} L_1(\mathbf{m}, \mathbf{u}, \mathbf{u}_1) = \frac{1}{2}\int_0^T \int_\Omega (R\mathbf{u} - \mathbf{d}^{obs})^2 dtdx \\ + \int_0^T \int_\Omega [(-M_1 \partial_t \mathbf{u}_1)^* \mathbf{u} + (M_2 \mathbf{u}_1)^* \mathbf{u} - (Q\mathbf{u}_1)^* \mathbf{u} - \mathbf{u}_1^* \mathbf{s}] dtdx \end{aligned} \tag{2.11}$$

## 2.2 Inverse problem

where

$$
\begin{aligned}
&M_1^* = M_1, (M_1\partial_t)^* = -M_1^*\partial_t = -M_1\partial_t, M_2^* = M_2 \\
&D^* = (D_i\partial_i)^* = -D_i^*\partial_i = D_i\partial_i = D \Rightarrow Q^* = Q
\end{aligned}
\tag{2.12}
$$

which is derived from the initial, final condition and boundary condition (eq. 2.6 and 2.10) (Yang et al., 2016). Derivative of $L_1$ with respect to wavefields $\mathbf{u}$:

$$
\frac{\partial L_1}{\partial \mathbf{u}} = \int_0^T \int_\Omega \Big( (R\mathbf{u} - \mathbf{d}^{obs})R - M_1\partial_t\mathbf{u}_1 + M_2\mathbf{u}_1 - Q\mathbf{u}_1 \Big)^* dtdx \tag{2.13}
$$

To satisfy the final condition of $\mathbf{u}_1$ (eq. 2.10), the time needs to be reversed by substituting $t' = T - t$ ($\partial_{t'} = -\partial_t$ and $dt' = -dt$) in the above equation. Thus we get

$$
\frac{\partial L_1}{\partial \mathbf{u}} = -\int_T^0 \int_\Omega \Big( R^*(R\mathbf{u} - \mathbf{d}^{obs}) + M_1\partial_{t'}\mathbf{u}_1 + M_2\mathbf{u}_1 - Q\mathbf{u}_1 \Big)^* dt'dx \tag{2.14}
$$

## 2.2 Inverse problem

By making $\frac{\partial L_1}{\partial \mathbf{u}} = 0$, we obtain a self-adjoint equation:

$$\begin{aligned} M_1 \partial_{t'} \mathbf{u}_1 + M_2 \mathbf{u}_1 - Q \mathbf{u}_1 &= -R^*(R\mathbf{u} - \mathbf{d}^{obs}) \\ \mathbf{u}_1 = (\mathbf{H}_1, \mathbf{E}_1)^T &= (H_{1x}, H_{1y}, H_{1z}, E_{1x}, E_{1y}, E_{1z})^T \end{aligned} \tag{2.15}$$

Here the wavefield residual $R^*(R\mathbf{u} - \mathbf{d}^{obs})$ is used as the sources for back propagation. The 1st order adjoint-state equation is:

$$\mathbf{u}_1 : \begin{cases} -\mu \frac{\partial \mathbf{H}_1}{\partial t'} - \nabla \times \mathbf{E}_1 = -R^*(R\mathbf{H} - \mathbf{H}^{obs}) \\ \varepsilon \frac{\partial \mathbf{E}_1}{\partial t'} + \sigma \mathbf{E}_1 - \nabla \times \mathbf{H}_1 = -R^*(R\mathbf{E} - \mathbf{E}^{obs}) \end{cases} \tag{2.16}$$

Derivative with respect to model parameters $\mathbf{m}$ from Eq. 2.9:

$$\nabla \Phi_1(\mathbf{m}) = \frac{\partial \Phi_1}{\partial \mathbf{m}} = \frac{\partial L_1}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{m}} + \frac{\partial L_1}{\partial \mathbf{m}} = \frac{\partial L_1}{\partial \mathbf{m}} = \int_0^T \mathbf{u}_1^* \left( \frac{\partial M_1}{\partial \mathbf{m}} \partial_t \mathbf{u} + \frac{\partial M_2}{\partial \mathbf{m}} \mathbf{u} \right) dt \tag{2.17}$$

## 2.2 Inverse problem

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The gradient of the objective function $\Phi_1(\mathbf{m})$ with respect to electric parameters are shown as:

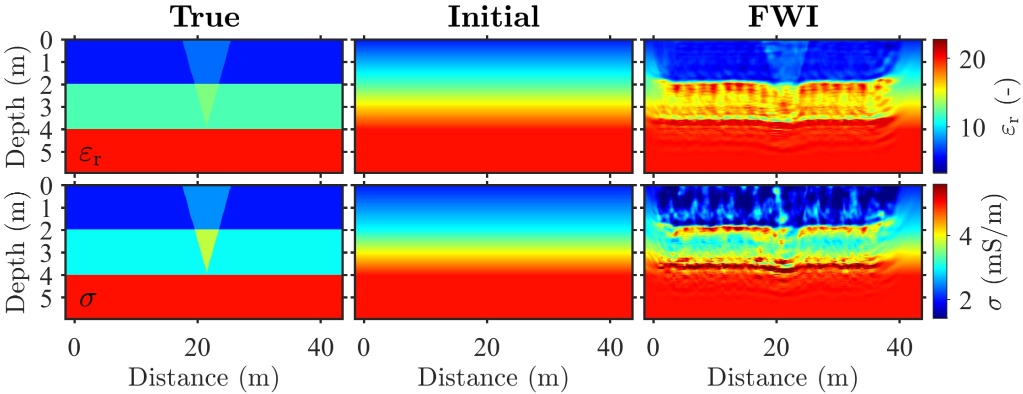
$$
\begin{aligned}
\nabla\Phi_1(\varepsilon) &= \int_0^T \left(E_{1x}\frac{\partial E_x}{\partial t} + E_{1y}\frac{\partial E_y}{\partial t} + E_{1z}\frac{\partial E_z}{\partial t}\right)dt \\
\nabla\Phi_1(\sigma) &= \int_0^T \left(E_{1x}E_x + E_{1y}E_y + E_{1z}E_z\right)dt
\end{aligned}
\tag{2.18}
$$

The magnetic permeability $\mu$ is assumed to be constant in the application of GPR. Logarithmic parameterization (especially for the media with strong conductivity variations) (Meles, 2011):

$$
\begin{aligned}
\tilde{\varepsilon} &= \log(\varepsilon/\varepsilon_0), \quad \nabla\Phi_1(\tilde{\varepsilon}) = \varepsilon\nabla\Phi_1(\varepsilon) \\
\tilde{\sigma} &= \log(\sigma/\sigma_0), \quad \nabla\Phi_1(\tilde{\sigma}) = \sigma\nabla\Phi_1(\sigma)
\end{aligned}
\tag{2.19}
$$

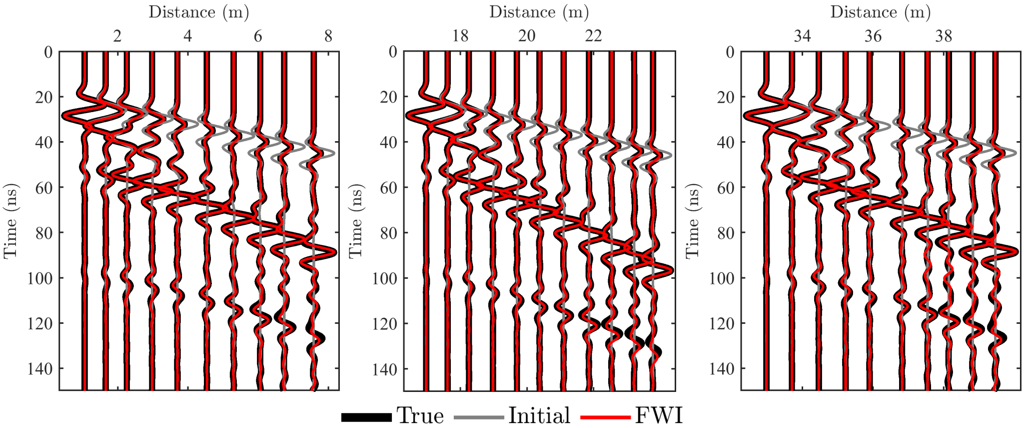
# Outline

# 3 Results



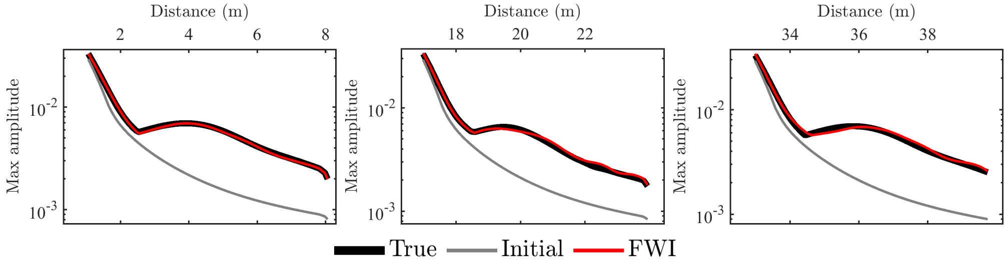

True model, initial model and the model reconstructed by GPR FWI (permittivity and conductivity simultaneously).

## 3 Results



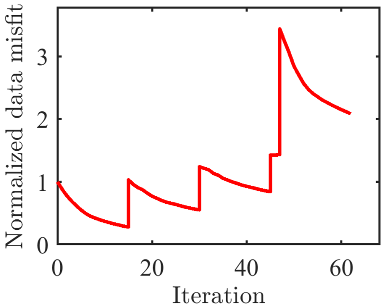

Waveform fit of the 1st, 9th and 17th shots.

# 3 Results




Maximum amplitude fit of the 1st, 9th and 17th shots.

# 3 Results




Misfit convergence curve as a function of iteration. A frequency-band variation from 5 to 40, 60, 80, 100 and 120 MHz is used in multiscale strategy to avoid cycle skipping in FWI (Bunks et al., 1995). The "jump" of misfit means that FWI switchs to the next inversion stage.

# Outline

# 4 Summary

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### Summary

(1) GPR simulation based on Maxwell's equations.

(2) GPR FWI derived from the adjoint-state method.

### Challenge

(1) Weak responds of the conductivity in surface-based GPR data.

Possible solution: joint inversion with electric resistance data.

(2) Cross-talk between permittivity and conductivity.

Possible solution: the 2nd order optimization method, such as truncated Newton method.

# References


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# Thank you for your attention

tan.qin@kit.edu

http://www.gpi.kit.edu/